COLLOQUIUM MATHEMATICUM

VOL. LX/LXI DÉDIÉ À M. ANTONI ZYGMUND 1990

CONIC DEGENERATION OF THE DIRAC OPERATOR

BY

R. SEELEY (BOSTON, MASSACHUSETTS)

0. Introduction. Consider a manifold M with an embedded hypersurface N. We obtain a family of Riemannian manifolds by shrinking N to a point, in such a way that the limiting manifold has a conic singularity with two copies of N as cross section. In this situation one can ask: What happens to the basic geometric operators in this limit? The case of $\overline{\partial}$ on a Riemann surface was treated in [SS]. Here, we consider the Dirac operator; a closely related paper [S] treats the Gauss–Bonnet operator.

The "shrinking" process is the following. Take a fixed metric $\widetilde{g}$ on N, and let x, $|x| < 1$, be a coordinate transversal to N. We define a family of metrics g_t on M, depending on the parameter t, by

$$g_t = dx^2 + (x^2 + t^2)\widetilde{g}, \quad |x| < 1. \tag{0.1}$$

For $|x| \geq 1$, the metric is extended in some convenient way. In the limit as $t \to 0$, this is a conic metric; the cross section of the cone is two copies of N. Denote by D_t the full, selfadjoint Dirac operator on M, with the metric g_t; by $D_{\max}$ the maximal realization of the Dirac operator on the conic limiting manifold; and by $\widetilde{D}$ the Dirac operator on N. As shown in [C], and at the end of Sec. 1 below, each eigenvalue μ of $\widetilde{D}$ with $|\mu| < 1/2$ contributes sections in the domain of $D_{\max}$ with singularities like $|x|^{-(n/2)\pm\mu}$, which are therefore in L^2. When such sections are present, D is not essentially selfadjoint, and there are various possible closed realizations (see [C] and [BS]). Our limit picks out a particular one.

THEOREM 1. *The graph of D_t has a limit as $t \to 0$, which is the graph of a selfadjoint Dirac operator D_0 on the conic limiting manifold. The domain of D_0 consists of the sections in the domain of $D_{\max}$ which satisfy the condition*

$$\lim_{x\to 0^+} |x|^{n/2} s(x) = \lim_{x\to 0^-} |x|^{n/2} s(x). \tag{0.2}$$

The limits in (0.2) are taken in $L^2(N)$. The existence of these limits means that the singularities $|x|^{-(n/2)-|\mu|}$ are required to vanish for $\mu \neq 0$; sections where the limits in (0.2) are finite and nonzero arise only from the harmonic spinors on N, with eigenvalue $\mu = 0$. Thus, when no such harmonic spinors exist, the link between the two sides of N dissolves completely; in the limit, there is no condition relating the sections on one side of N to sections on the other side.

When N is the standard unit sphere, then the limiting metric g_0 is the standard metric on the unit ball; so functions in the maximal domain are in the Sobolev space H^1, ruling out any singularities $|x|^{-(n/2)\pm|\mu|}$ for $|\mu| < 1/2$. This gives a peculiar proof that the Dirac operator on the standard unit sphere has no eigenvalue with $|\mu| < 1/2$. (In the case of S^1, we have to specify the spin structure; it is the nontrivial one arising by considering S^1 as the boundary of the unit 2-disk.)

Now suppose that $\dim(M)$ is even, and let D_t^+ denote the Dirac operator D_t restricted to the "+ spinors". The index of D_t^+ is the $\widehat{A}$ genus of M. Now each eigenvalue of $\widetilde{D}$ with $|\mu| < 1/2$ contributes on each side of N just one section in the domain of $D_{\max}$, with singularity $x^{-(n/2)+\mu}$ as $x \to 0^+$ and $|x|^{-(n/2)-\mu}$ as $x \to 0^-$ (see end of Sec. 1).

THEOREM 2. *The graph of D_t^+ has a limit as $t \to 0$, which is the graph of a Dirac operator D_0^+ on the conic limiting manifold. The domain of D_0^+ is again determined by the condition* (0.2).

Since graph continuity preserves the index (see [SS]), we obtain:

COROLLARY 1. *The limiting operator D_0^+ has index* $\operatorname{ind}(D_0^+) = \widehat{A}(M)$.

Suppose now that N separates M into two disjoint parts M_+ and M_-. Then we obtain two disjoint Riemannian manifolds $\overline{M}_+$ and $\overline{M}_-$, each with a conic singularity with cross section N. On each of these, we can consider Dirac operators with appropriate domains. Denote by $D^+_{\mathrm{Dir}}(\overline{M}_\pm)$ the restriction of $D^+_{\max}(\overline{M}_\pm)$ to sections with

$$|x|^{n/2}s(x) = o(1),$$

and by $D^+_{\mathrm{Neu}}(\overline{M}_\pm)$ the restriction to sections with $o(1)$ replaced by $O(1)$. The distinction between these two realizations lies in the treatment of the harmonic spinors, which give rise to sections where the limits in (0.2) are finite and nonzero; the Dirichlet domain requires these limits to vanish, while the Neumann domain leaves them free. Denote the indices of these realizations by $\widehat{A}_{\mathrm{Dir}}(\overline{M}_\pm)$ and $\widehat{A}_{\mathrm{Neu}}(\overline{M}_\pm)$. Then Theorem 2 yields:

COROLLARY 2. *In the above situation*

$$\widehat{A}(M) = \widehat{A}_{\mathrm{Dir}}(\overline{M}_+) + \widehat{A}_{\mathrm{Neu}}(\overline{M}_-) = \widehat{A}_{\mathrm{Dir}}(\overline{M}_+) + \widehat{A}_{\mathrm{Dir}}(\overline{M}_-) + \nu(\widetilde{D})$$

where $\nu(\widetilde{D})$ is the nullity of the Dirac operator on N.

The outline of the paper is as follows. Section 1 reviews the separation of variables formula for the Dirac operator given by Chou [C]. Section 2 constructs a family of parametrices for D_t near the hypersurface N, and studies their limit as $t \to 0$. Section 3 reviews the argument for graph continuity from [SS], thus proving Theorems 1 and 2. In the process, we include a part of the argument that was regrettably omitted from [SS].

The author thanks Haynes Miller and I. M. Singer for helpful comments about $\widehat{A}(M)$ and the Dirac operator, and J. Bruening for a careful reading of an earlier draft.

1. The representations of D and D^+ using eigenvectors of $\widetilde{D}$. This section is essentially a review of part of [C].

A metric g on M gives an inner product on $T(M)$, and thus determines a Clifford bundle on M, a vector bundle whose fibre at any point p is the tensor algebra generated by the tangent vectors, reduced modulo the relations

$$\partial/\partial x_1 \otimes \partial/\partial x_2 + \partial/\partial x_2 \otimes \partial/\partial x_1 = -2\langle \partial/\partial x_1, \partial/\partial x_2\rangle_g . \tag{1.1}$$

We denote by a $\cdot$ the multiplication in this Clifford bundle. If M is a spin manifold, it carries a spin bundle S on which the Clifford bundle acts. The metric g defines a connection on S. The Dirac operator is then expressed, using an orthonormal frame $\{e_j\}$ for $T(M)$, as

$$D = \sum e_j \cdot \nabla_{e_j} . \tag{1.2}$$

With our local representation near N, we can choose an orthonormal frame $\{e_1, \ldots, e_n\}$ for N, and supplement it with $e_0 = \partial/\partial x$.

When $\dim(M) = n+1$ is even, then S is a direct sum of subbundles $S^\pm$ determined by the action of the volume element

$$\omega_{n+1} = i^{(n+1)/2} \frac{\partial}{\partial x} \cdot \omega_n , \quad \omega_n = e_1 \cdot \ldots \cdot e_n .$$

Since $\omega_{n+1} \cdot \omega_{n+1} = 1$, subbundles $S^\pm$ of S can be (and are) defined by

$$\omega_{n+1} \cdot s^\pm = \pm s^\pm \quad \text{for any section } s^\pm \text{ in } S^\pm.$$

This gives a restriction of D,

$$D^+ : C^\infty(S^+) \to C^\infty(S^-) .$$

The restriction $S^+|_N$ determines a spin bundle $S(N)$ on N, and as in [C], we denote by $\widetilde{D}$ the Dirac operator on this $S(N)$. $\widetilde{D}$ is selfadjoint elliptic, so has an orthonormal basis of eigenfunctions φ_j with eigenvalues μ_j. Denote by $\overline{\varphi}_j$ the parallel translation of φ_j along the curves with tangents $\partial/\partial x$. Consider a metric of the form

$$dx^2 + h(x)^2 \widetilde{g} \tag{1.3}$$

where $\widetilde{g}$ is the metric on N, and $h(x)$ a warping factor; in (0.1), $h(x) = (x^2 + t^2)^{1/2}$. According to [C, Prop. (2.5)], if $\widetilde{D}\varphi = \mu\varphi$, $\overline{\varphi}$ is the parallel translation of φ, and f and g are functions of x alone, then

$$D\left(f\overline{\varphi}+g\frac{\partial}{\partial x}\cdot\overline{\varphi}\right)=\left(f'+\frac{n}{2}\frac{h'}{h}f-\frac{\mu}{h}g\right)\frac{\partial}{\partial x}\cdot\overline{\varphi}+\left(\frac{\mu}{h}f-g'-\frac{n}{2}\frac{h'}{h}g\right)\overline{\varphi}$$

where $n = \dim(N)$. Moreover, the norm of any section $f\overline{\varphi} + g\partial/\partial x\cdot\overline{\varphi}$ supported in $\{|x| < 1\}$ is

$$\|\varphi\|^2\int_{-1}^{1}(|f|^2+|g|^2)h^n\,dx\,.$$

So if we introduce a normalizing factor $h^{-n/2}$ and represent a section as

$$s=\sum h^{-n/2}\left(f_j\overline{\varphi}_j+g_j\frac{\partial}{\partial x}\cdot\overline{\varphi}_j\right) \tag{1.4}$$

then

$$Ds=\sum h^{-n/2}\left[(h^{-1}\mu_j f_j-g_j')\overline{\varphi}_j+(f_j'-h^{-1}\mu_j g_j)\frac{\partial}{\partial x}\cdot\overline{\varphi}_j\right].$$

This gives the desired representation of D near the hypersurface N; it is the direct sum of operators of the form

$$\begin{aligned}&\begin{bmatrix}0&-1\\1&0\end{bmatrix}\frac{\partial}{\partial x}+\frac{1}{h}\begin{bmatrix}\mu_j&0\\0&-\mu_j\end{bmatrix}\\&\qquad=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\left(\frac{\partial}{\partial x}+\frac{1}{h}\begin{bmatrix}0&-\mu_j\\-\mu_j&0\end{bmatrix}\right)\end{aligned}\tag{1.5}$$

acting on pairs $\begin{bmatrix}f\\g\end{bmatrix}$ with norm $\int_{-1}^{1}(|f(x)|^2+|g(x)|^2)\,dx$. The pair $\begin{bmatrix}f_j\\g_j\end{bmatrix}$ represents $h^{-n/2}(f_j\overline{\varphi}_j+g_j\partial/\partial x\cdot\overline{\varphi}_j)$, and $\widetilde{D}\varphi_j=\mu_j\varphi_j$.

To represent $D^+ : C^\infty(S^+)\to C^\infty(S^-)$, we use the convenient fact that

$$\omega_{n+1}\cdot\left(\overline{\varphi}\pm\frac{\partial}{\partial x}\cdot\overline{\varphi}\right)=\pm\overline{\varphi}+\frac{\partial}{\partial x}\cdot\overline{\varphi}.$$

Thus $S^\pm$ is spanned by sections $\{f_j(x)(\overline{\varphi}_j\pm\partial/\partial x\cdot\overline{\varphi}_j)\}$, and D^+ is a direct sum of operators (1.5) acting on pairs $\begin{bmatrix}f\\f\end{bmatrix}$, which is equivalent to $\partial/\partial x - h^{-1}\mu_j$ acting on f.

In the limiting case, where $h(x) = |x|$, the analysis in [BS] shows that each eigenvalue μ_j in $(-1/2, 1/2)$ contributes a function f_j with a singularity x^{μ_j} as $x\to 0^+$, and $|x|^{-\mu_j}$ as $x\to 0^-$, yielding a section s_j in the maximal domain of D^+ with singularity $x^{-(n/2)+\mu_j}$ as $x\to 0^+$ and $|x|^{-(n/2)-\mu_j}$ as $x\to 0^-$. For the full Dirac operator in (1.5), the eigenvalues are $\pm\mu_j$, so there is a section on each side of N with singularity $|x|^{-(n/2)\pm\mu_j}$.

2. A parametrix for D_t near N. To construct a parametrix for the full Dirac operator D_t, we cancel the matrix $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ in (1.5), introduce eigenvectors $\left[\begin{smallmatrix} 1 \\ \pm 1 \end{smallmatrix}\right]$ for the matrix $\left[\begin{smallmatrix} 0 & -\mu \\ -\mu & 0 \end{smallmatrix}\right]$, and thus obtain operators of the form

$$\frac{\partial}{\partial x} \mp h^{-1}\mu . \tag{2.1}$$

So consider the equation

$$u'(x) - \mu h_t(x)^{-1}u(x) = f(x) \tag{2.2}$$

with f and u in L^2, and $h_t(x) = (x^2 + t^2)^{1/2}$. The solution is

$$u(x) = (x + h_t(x))^{\mu}\Big[\int_{?}^{x} (y + h_t(y))^{-\mu} f(y)\, dy + c\Big] \tag{2.3}$$

where the questionable limit of integration is to be determined by our desire for a limit as $t \to 0$. Note that $x + h_t(x)$ is a positive and increasing function for real x. Thus in (2.3) we would like $x < y$ when $\mu > 0$, and $y < x$ when $\mu < 0$. So we define a parametrix for (2.2) on the interval $(-1, 1)$ by

$$Q_t f(x) = \begin{cases} (x + h_t(x))^{\mu} \int_1^x (y + h_t(y))^{-\mu} f(y)\, dy & \text{if } \mu > 0, \\ (x + h_t(x))^{\mu} \int_{-1}^x (y + h_t(y))^{-\mu} f(y)\, dy & \text{if } \mu \le 0. \end{cases} \tag{2.4}$$

We will show that this has a limit as $t \to 0$. For the case $\mu > 0$ the limit is

$$Q_0 f(x) = \begin{cases} \int_1^x (x/y)^{\mu} f(y)\, dy, & x > 0, \\ \int_0^x |y/x|^{\mu} f(y)\, dy, & x < 0, \end{cases} \qquad \mu > 0 . \tag{2.5}$$

For $\mu < 0$ the lower limits are 0 for $x > 0$ and -1 for $x < 0$; for $\mu = 0$, we have merely $Q_0 f(x) = \int_{-1}^x f(y)\, dy$. (With $\mu = 0$, we could just as well integrate from 1 as from -1.)

LEMMA 1. *For $t < 1$,*

$$\|Q_t\| \le C(1 + |\mu|)^{-1} , \tag{2.6}$$

$$\|Q_t - Q_0\| \le C(1 + |\mu|)^{-1}(t + t^{\mu}) \log(1 + 1/t) , \tag{2.7}$$

with C independent of μ and t.

Proof of (2.6). The case $\mu = 0$ is clear, since then Q_t is independent of t; and the operator Q_t for $\mu < 0$ is minus the adjoint of the one for $-\mu$; so it suffices to take $\mu > 0$. For this case, we split both domain and range as the direct sum of $L^2(-1, 0)$ and $L^2(0, 1)$.

For $0 \le x \le y$ we have

$$\frac{x + h_t(x)}{y + h_t(y)} \le \frac{x + t}{y + t} . \tag{2.8}$$

So the integral operator on $L^2(0,1)$ with kernel

$$(x+h_t(x))^\mu (y+h_t(y))^{-\mu}, \quad 0 \le x \le y \le 1,$$

has norm bounded by

$$\text{(2.9)} \quad \sup_{x\le 1}(x+t)^\mu \int_x^1 (y+t)^{-\mu}\,dy + \sup_{y\le 1}(y+t)^{-\mu}\int_0^y (x+t)^\mu\,dx \le \frac{C}{1+\mu}$$

with C independent of μ and t, $0 \le t \le 1$. Likewise, the operator from $L^2(0,1)$ to $L^2(-1,0)$ defined by

$$Kf(x) = (x+h_t(x))^\mu \int_0^1 (y+h_t(y))^{-\mu} f(y)\,dy, \quad -1 \le x \le 0,$$

has norm

$$\text{(2.10)} \quad \begin{aligned} \|K\| &\le \Big[\int_{-1}^0 (x+h_t(x))^{2\mu}\,dx \int_0^1 (y+h_t(y))^{-2\mu}\,dy\Big]^{1/2} \\ &= t^{2\mu}\int_0^1 (y+h_t(y))^{-2\mu}\,dy \le t^{2\mu}\int_0^1 (y+t)^{-2\mu}\,dy \\ &= \frac{t}{2\mu-1}\left[1-\left(\frac{t}{1+t}\right)^{2\mu-1}\right]. \end{aligned}$$

(In the integral from -1 to 0, replace x by $-y$.) When $\mu \ge 1$ this is $\le Ct/(1+\mu)$. When $\mu < 1$, the Mean Value Theorem applied to $\left(\frac{t}{1+t}\right)^x$ gives $\|K\| \le t\log(1+1/t)$. Thus, for $0 \le t \le 1$,

$$\text{(2.11)} \quad \|K\| \le \frac{C}{1+\mu} t\log(1+1/t).$$

This gives the desired estimate for Q_t from $L^2(0,1)$ to $L^2(-1,1)$. The part of Q_t acting from $L^2(-1,0)$ to $L^2(0,1)$ is 0, and the part from $L^2(-1,0)$ to $L^2(-1,0)$ is equivalent, by an obvious change of variable, to an operator on $L^2(0,1)$ with kernel

$$(x+h_t(x))^{-\mu}(y+h_t(y))^\mu, \quad 0 \le y \le x \le 1.$$

This is the adjoint of the one estimated in (2.9), so has the same norm.

P r o o f o f (2.7). As before, it suffices to take $\mu > 0$. In view of (2.11), and the adjoint relation exploited at the end of the previous proof, it suffices to consider the operator on $L^2(0,1)$ with kernel

$$\left[\frac{x+h_t(x)}{y+h_t(y)}\right]^\mu - \left[\frac{x}{y}\right]^\mu, \quad 0 \le x \le y \le 1.$$

We must show that this operator satisfies the estimate (2.7). For brevity, we drop the subscript t on h_t. Using (2.8) again, it suffices to treat the kernel

$$k(x,y) = \left(\frac{x+t}{y+t}\right)^{\mu} - \left(\frac{x}{y}\right)^{\mu}, \quad 0 \le x \le y \le 1.$$

By Schur's Test, the norm of this operator is dominated by

$$\sup_{x \le 1} \int_x^1 k(x,y)\,dy + \sup_{y \le 1} \int_0^y k(x,y)\,dx$$

$$= \frac{1}{1-\mu}[t^{\mu}(1+t)^{1-\mu} - t] + \frac{t}{\mu+1}\left[1 - \left(\frac{t}{1+t}\right)^{\mu}\right], \quad \mu \neq 1.$$

The second term is $\le t/(\mu+1)$. The first is $\le t/(1-\mu)$ for $\mu > 1$, and the Mean Value Theorem shows that it is $\le t^{\min(\mu,1)}\log(1+1/t)$. This proves (2.7), and with it Lemma 1.

3. The global parametrix, and graph continuity. This section is essentially a review of the corresponding part of [SS], *mutatis mutandis.*

Since D_t is selfadjoint, $i + D_t$ is invertible. We construct the resolvent $(i+D_t)^{-1}$ and show that it converges in norm as $t \to 0$, thus proving graph continuity of the family D_t at $t = 0$.

Let Q_i be a parametrix for the Dirac operator D_t on the part of M where $|x| > 1/2$, the "interior" of M. Since the metric (0.1) varies with t, so will Q_i vary with t; but it is easy to guarantee that Q_i varies smoothly in t, for $t \ge 0$. Thus

$$D_t Q_i = I + T_i \tag{3.1}$$

where the kernel of T_i is C^{∞} on $M \times M \times [0,1]$, and the kernel of Q_i is C^{∞} off the diagonal. The parametrices Q_t in (2.4), acting in the various eigenspaces, can be combined to give a parametrix for D_t in $\{|x| < 1\}$; we denote this, too, by Q_t. Each operator in (2.4) is compact, and the direct sum of these operators converges in norm, because of the factor $(1+\mu)^{-1}$ in (2.6), so the combined operator Q_t is also compact.

We now set

$$P_t = \varphi_i Q_i \psi_i + \varphi_c Q_t \psi_c \tag{3.2}$$

where $\psi_i + \psi_c = 1$, $\varphi_i\psi_i = \psi_i$, $\varphi_c\psi_c = \psi_c$, while φ_i and ψ_i vanish for $|x| < 1/2$, and φ_c and ψ_c are supported in $\{|x| < 1\}$. We have

$$(i + D_t)P_t = I + R_t \tag{3.3}$$

where the compact remainder is

$$R_t = iP_t + \varphi_i T_i \psi_i + \varphi_i' e_0 \cdot Q_i \psi_i + \varphi_c' e_0 \cdot Q_t \psi_c$$

with e_0 denoting Clifford multiplication by $\partial/\partial x$. Since $I + R_t$ is Fredholm with index zero, and $i + D_t$ is invertible, it follows that P_t is a Fredholm operator, with index zero, from $L^2(S)$ to the domain of D_t. Using the representation (1.4), we can think of P_t as acting in a fixed Hilbert space, independent of t. Define D_0 to be the Dirac operator D in the conic limiting metric $g_0 = dx^2 + x^2\widetilde{g}$, with domain defined by

$$\lim_{x \to 0+} |x|^{n/2} s(x) = \lim_{x \to 0-} |x|^{n/2} s(x), \tag{3.4}$$

both limits existing (as in (0.2)). One checks that both P_0 and P_0^* map into this domain; in eigenspaces with $\mu \neq 0$ both limits are 0, while in the 0 eigenspace they are equal. Moreover, (3.4) defines a selfadjoint realization of D, and (3.3) holds true for $t = 0$.

Let N_0 be the nullspace of P_0, and $k = \dim(N_0)$. Since P_0 has index zero as a map into $\mathrm{Dom}(D_0)$, there is a k-dimensional subspace, V, of $\mathrm{Dom}(D_0)$, which is linearly independent of $\mathrm{Range}(P_0)$. We will show in Lemma 3 below that V can be taken as a subspace of

$$C_c^\infty(S) = \text{sections of } S \text{ vanishing in a neighborhood of } N.$$

(This is the argument omitted from [SS].) Let V_0 map N_0 isomorphically onto V, and $N_0^\perp$ to 0. Then $P_0 + V_0$ is an isomorphism of $L^2(S)$ onto $\mathrm{Dom}(D_0)$, so

$$(i + D_t)(P_t + V_0) = I + R_t + (i + D_t)V_0$$

is invertible when $t = 0$. Since V_0 is in $C_c^\infty(S)$, the right hand side varies continuously with t, and

$$(i + D_t)^{-1} = (P_t + V_0)(I + R_t + (i + D_t)V_0)^{-1} \tag{3.5}$$

varies continuously with t, as $t \to 0$. This proves Theorem 1.

Now suppose that $\dim(M)$ is even, so there is a direct sum $S = S^+ \oplus S^-$, with D_t mapping $C^\infty(S^\pm)$ to $C^\infty(S^\mp)$. Denote by D_t^+ the restriction of D_t to the sections of S^+. We can represent the full Dirac operator D_t as $\begin{bmatrix} 0 & D_t^+ \\ D_t^- & 0 \end{bmatrix}$, with $D_t^- = (D_t^+)^*$. Hence the continuity of $(i + D_t)^{-1}$ implies that of $\begin{bmatrix} I & iD_t^+ \\ -(iD_t^+)^* & I \end{bmatrix}^{-1}$, and hence the graph continuity of D_t^+ (see [SS]). This proves Theorem 2, and with it Corollary 1.

Now consider the situation of Corollary 2 of Theorem 2, where N divides M into two disjoint parts; denote the resulting conic limit manifolds by $\overline{M}_+$ and $\overline{M}_-$. Denote by $\nu(\widetilde{D})$ the nullity of the Dirac operator $\widetilde{D}$ on N. Then the condition (3.4) defining the domain of the conic limiting operator involves $\nu(\widetilde{D})$ matching conditions. These can be replaced by an equal number of vanishing conditions, by requiring

$$\lim_{x\to 0+} |x|^{n/2}s(x) = 0\,, \qquad \lim_{x\to 0-} |x|^{n/2}s(x) \quad \text{exists.}$$

Since the number of conditions is the same, the index is the same. Moreover, these conditions define precisely the Dirichlet realization on $\overline{M}_+$ and the Neumann realization on $\overline{M}_-$. Hence

$$\begin{aligned}\widehat{A}(M) &= \operatorname{ind}(D_0) = \operatorname{ind}(D^+_{\mathrm{Dir}}(\overline{M}_+)) + \operatorname{ind}(D^+_{\mathrm{Neu}}(\overline{M}_-)) \\ &= \widehat{A}_{\mathrm{Dir}}(\overline{M}_+) + \widehat{A}_{\mathrm{Neu}}(\overline{M}_-)\,.\end{aligned}$$

If we replace the condition "$\lim_{x\to 0-} |x|^{n/2}s(x)$ exists" by the condition "$\lim_{x\to 0-} |x|^{n/2}s(x) = 0$", we add $\nu(\widetilde{D})$ boundary conditions, and thus decrease the index by $\nu(\widetilde{D})$. Thus we obtain the final statement of Corollary 2,

$$\widehat{A}(M) = \widehat{A}_{\mathrm{Dir}}(\overline{M}_+) + \widehat{A}_{\mathrm{Dir}}(\overline{M}_-) + \nu(\widetilde{D})\,.$$

It remains to prove that the complementary space V above can be taken in $C^\infty_c(S)$. This depends on two lemmas.

LEMMA 2. *If u is in* $\operatorname{Dom}(D_0)$ *and orthogonal (in the graph norm for D_0) to both* $\operatorname{Range}(P_0)$ *and $C^\infty_c(S)$, then $u = 0$.*

Proof. The conditions on u are

$$(u, v) + (D_0u, D_0v) = 0 \quad \text{for all } v \text{ in } C^\infty_c(S)\,, \tag{3.6}$$

$$(u, P_0f) + (D_0u, D_0P_0f) = 0 \quad \text{for all } f \text{ in } L^2(S)\,. \tag{3.7}$$

The first condition shows that $u + D_0^2u = 0$ except perhaps on N, so D_0u is in the maximal domain of D. We will show that D_0u is actually in the domain of D_0, so the equation $u + D_0^2u = 0$ implies that $u = 0$.

For f supported in $\{|x| < 1/2\}$, $D_0P_0f = f$, so the second condition (3.7) gives

$$D_0u = -P_0^*u \quad \text{in } \{|x| < 1/2\}\,. \tag{3.8}$$

But (like P_0) P_0^* maps into the domain of D_0. Since the domain of D_0 is defined by a limiting condition as $x \to 0^\pm$, it follows from (3.8) that D_0u is in the domain of D_0, and this completes the proof of Lemma 2.

LEMMA 3. $\operatorname{Range}(P_0)$ *has a linear complement spanned by elements of* $C^\infty_c(S)$.

Proof. By Lemma 2, $\operatorname{Range}(P_0) + C^\infty_c(S)$ is dense in $\operatorname{Dom}(D_0)$. But $\operatorname{Range}(P_0)$ has finite codimension in that space, so $\operatorname{Range}(P_0) + C^\infty_c(S)$ is also a closed subspace, and thus gives all of $\operatorname{Dom}(D_0)$. Now if $\{u_1, \ldots, u_k\}$ span a linear complement of $\operatorname{Range}(P_0)$ in $\operatorname{Dom}(D_0)$, then each $u_j = P_0f_j + v_j$ for a v_j in $C^\infty_c(S)$; then the v_j span a linear complement of $\operatorname{Range}(P_0)$.

REFERENCES

[BS] J. Bruening and R. Seeley, *An index theorem for first order regular singular operators*, Amer. J. Math. 110 (1988), 659–714.

[C] A. W. Chou, *The Dirac operator on spaces with conical singularities and positive scalar curvatures*, Trans. Amer. Math. Soc. 289 (1985), 1–40.

[S] R. Seeley, *Conic degeneration of the Gauss–Bonnet operator*, preprint.

[SS] R. Seeley and I. M. Singer, *Extending* $\bar{\partial}$ *to singular Riemann surfaces*, J. Geom. Phys. 5 (1988), 121–136.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MASSACHUSETTS AT BOSTON
HARBOR CAMPUS
BOSTON, MASSACHUSETTS 02125–3393, U.S.A.

Reçu par la Rédaction le 25.4.1990