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ON DIFFERENTIAL OPERATORS OF SECOND ORDER ON RIEMANNIAN MANIFOLDS WITH NONPOSITIVE CURVATURE

BY

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Introduction. Let $G|H$ be a Riemannian homogeneous space, where G is a connected Lie group and $H \subset G$ is a closed subgroup; the action of G is assumed to be effective. A differential operator D on $G|H$ is called *invariant* if

$$D(f\cdot g)\cdot g^{-1} = Df$$

for any smooth function $f\colon G|H \to \boldsymbol{R}$ and any $g \in G$. A Riemannian symmetric space M may be written as a homogeneous space $G|H$ ([3], vol II, p. 224); let $\mathfrak{A}$ be the algebra of invariant differential operators. Then $\mathfrak{A}$ is commutative ([2], p. 396). From that result Sumitomo raised the following problem ([8], p. 132, P 807):

PROBLEM. Let M be a Riemannian homogeneous space with commutative algebra $\mathfrak{A}$. Is the Ricci tensor parallel?

The following theorem gives a partial answer:

THEOREM. *Let M be a Riemannian homogeneous space with nonpositive sectional curvature and commutative algebra $\mathfrak{A}$. Then the Ricci tensor is parallel.*

1. Differential operators of the second order. Let M be a connected Riemannian manifold, $\dim M \geqslant 2$; let g_{ij} be the components of the metric tensor g in local coordinates (u^i); let ∇ denote covariant differentiation. As usual, raising and lowering of indices are defined. Let Δ denote the Laplacian, $\Delta f = g^{ij}\nabla_j\nabla_i f$, $f\colon M \to \boldsymbol{R}$. Manifolds, maps, etc. are of class C^∞.

The following results are essential tools for the proof of the Theorem.

1.1. LEMMA (Lichnerowicz [4]). *Let M be a differentiable manifold with an affine connection ∇ without torsion. Any differential operator D*

of order r on M can be expressed by

$$Df = \sum_{p\leqslant r} a^{i_1\dots i_p}\nabla_{i_p}\dots\nabla_{i_1}f, \tag{1.1.1}$$

where $a^{i_1\dots i_p}$ *are the components of a contravariant symmetric tensor on* M; *moreover, this expression is unique.*

1.2. Lemma (Sumitomo [9], Theorem 2.2). *In order that a differential operator* D *of second order,* $D = a^{ij}\nabla_j\nabla_i$, *commute with the Laplacian it is necessary and sufficient that the coefficient tensor satisfy the following three equations:*

$$\nabla_k a_{ij} + \nabla_i a_{jk} + \nabla_j a_{ki} = 0, \tag{1.2.1}$$

$$-g^{rs}\nabla_r\nabla_s a_{ij} - 2a^{rs}R_{irsj} + R_i^r a_{rj} + R_j^r a_{ri} = 0, \tag{1.2.2}$$

$$a^{rs}\nabla_i R_{rs} - 2a^{rs}\nabla_r R_{si} = 0; \tag{1.2.3}$$

R_{hijk}, *resp.* R_{ij}, *are the components of the curvature tensor, resp. the Ricci ensor* (1).

1.3. Lemma (Sumitomo [9], Theorem 2.3). *Let* D *be a differential operator of second order on* M, $D = a^{ij}\nabla_j\nabla_i$. *If* $D\Delta = \Delta D$, *then each of the conditions*

(1.3.1) *M is compact,*

(1.3.2) *M is irreducible,*

$$\mathrm{rank}(R_{ij}) = n = \dim M \tag{1.3.3}$$

implies

$$\mathrm{trace}\, a_{ij} = \mathrm{const} \tag{1.3.4}$$

on M.

1.4. Lemma. *Let* A *be a differentiable symmetric* $(0,2)$*-tensor* (*with components* A_{ij}) *on* M *with*

$$\mathrm{trace}\, A = \mathrm{const}, \tag{1.4.1}$$

$$\nabla_k A_{ij} + \nabla_i A_{jk} + \nabla_j A_{ki} = 0. \tag{1.4.2}$$

Then

$$\tfrac{1}{2}\Delta A_{ij}A^{ij} = -2\sum_{i<j} K_{ij}(\lambda_i - \lambda_j)^2 + g^{rs}\nabla_r A_{ij}\nabla_s A^{ij}; \tag{1.4.3}$$

$\lambda_1, \dots, \lambda_n$ *are the eigenvalues of* A *with corresponding* (*orthonormal*) *eigenvectors* $E_1, \dots, E_n$ *and* K_{ij} *is the sectional curvature corresponding to* $\{E_i, E_j\}$.

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(1) The sign of the curvature tensor in [9] differs from that in [1], p. 30. We use the notation of Eisenhardt.

Proof. We have

$$\tag{1.4.4} \tfrac{1}{2}\Delta A_{ij}A^{ij} = g^{pq}(\nabla_q\nabla_p A_{ir})A^{ir} + g^{pq}\nabla_p A_{ir}\nabla_q A^{ir}.$$

Using (1.4.2) and the well-known Ricci identity ([1], p. 30, (11.16)) and the symmetry of A^{ir}, we get

$$\tag{1.4.5} \begin{aligned} -g^{pq}(\nabla_q\nabla_p A_{ir})A^{ir} &= g^{pq}(\nabla_p\nabla_i A_{rq} + \nabla_p\nabla_r A_{qi})A^{ir} \\ &= 2g^{pq}(\nabla_i\nabla_p A_{qr} + R^h{}_{qip}A_{rh} + R^h{}_{rip}A_{qh})A^{ir}. \end{aligned}$$

From (1.4.1) and (1.4.2) we have

$$\tag{1.4.6} g^{pq}\nabla_p A_{qi} = 0.$$

Let $p \in M$; we choose local coordinates (u^i) corresponding to $\{E_r\}_{r=1}^n$; then for $\{E_i, E_j\}$ $(i \neq j;\ i, j$ fixed)

$$\tag{1.4.7} K_{ij} = R_{ijji}.$$

(1.4.5)-(1.4.7) imply

$$\tag{1.4.8} -g^{pq}(\nabla_p\nabla_q A_{ir})A^{ir} = 2\sum_{i<j} K_{ij}(\lambda_i - \lambda_j)^2;$$

(1.4.3) follows from (1.4.4) and (1.4.8).

For an analogous formula for Codazzi tensors with constant trace compare [6] and [10], Corollary 1, Theorem 1.

1.5. COROLLARY. *Let M be a Riemannian manifold with nonpositive sectional curvature. Let A be a differentiable symmetric $(0,2)$-tensor on M which fulfills* (1.4.1) *and* (1.4.2). *If*

$$\tag{1.5.1} A_{ij}A^{ij} = \text{const},$$

then

$$\tag{1.5.2} \nabla_k A_{ij} = 0 \quad \text{on } M.$$

1.6. PROPOSITION. *Let M be a Riemannian manifold of nonpositive curvature and let D be a differential operator of second order on M,*

$$D = a^{ij}\nabla_i\nabla_j,$$

which commutes with the Laplacian. Then each of the conditions

$$\tag{1.6.1} a^i{}_i = \text{const} \quad \text{and} \quad a_{ij}a^{ij} = \text{const},$$

$$\tag{1.6.2} \operatorname{rank}(R_{ij}) = n = \dim M, \quad a_{ij}a^{ij} = \text{const}$$

implies

$$\tag{1.6.3} \nabla_k a_{ij} = 0.$$

Proof by 1.2-1.5.

The following lemma is a direct consequence of the de Rham decomposition theorem (cf. [2], p. 187); we owe it to R. Walden.

1.7. LEMMA. *Let M be a simply connected Riemannian manifold and A a symmetric covariant constant $(0,2)$-tensor on M. Then*

(1.7.1) *the eigenvalues $\lambda_1, \dots, \lambda_n$ of A are constant;*

(1.7.2) *the eigendistributions are integrable and parallel;*

(1.7.3) *M is the Riemannian product,*

$$M = M_0 \times M_1 \times \dots \times M_k,$$

of the integral manifolds M_β $(\beta = 0, \dots, k)$ of the eigendistributions; each integral manifold is a totally geodesic submanifold of M.

1.8. REMARK (cf. [7], § 1). If M is irreducible and simply connected, $\nabla A = 0$ implies $A_{ij} = \lambda g_{ij}$, $\lambda = \text{const}$; if M is reducible and if $A(\beta)$, resp. $g(\beta)$, denote the tensors induced by A, resp. g, on M_β, then, if M_β is irreducible, $\nabla A = 0$ implies

(1.8.1) $$A(\beta) = \lambda_\beta g(\beta), \quad \beta = 0, \dots, k;$$

λ_β are the eigenvalues of A.

1.9. COROLLARY. *Let M be a Riemannian manifold with nonpositive sectional curvature. Let A be a differentiable symmetric $(0,2)$-tensor on M, $A_{ij} \not\equiv v g_{ij}$, $v\colon M \to \boldsymbol{R}$, which fulfills*

(1.9.1) $$\nabla_k A_{ij} + \nabla_i A_{jk} + \nabla_j A_{ki} = 0,$$

(1.9.2) $$\operatorname{trace} A = \text{const}.$$

Then locally we have a decomposition of M (compare (1.8)-(1.9)) *and for each irreducible factor M_β we have*

(1.9.3) $$A(\beta) = a_\beta g(\beta), \quad \beta = 0, \dots, k.$$

1.10. THEOREM. *Let M be a simply connected Riemannian manifold with nonpositive sectional curvature. Let D be a differential operator of second order,*

$$D = a^{ij} \nabla_j \nabla_i .$$

Let

(1.10.1) *M be irreducible,*

(1.10.2) $$a^{ij} a_{ij} = \text{const}.$$

Then D commutes with the Laplacian Δ iff $D = \lambda \Delta$, $\lambda = \text{const}$.

If M is reducible, D induces differential operators $D(\beta)$ of second order on $M(\beta)$; let $\Delta(\beta)$ be the Laplacian of $M(\beta)$. Then we have

1.11. THEOREM. *Let M be a simply connected Riemannian manifold of nonpositive sectional curvature. If there exists a differential operator D of second order on M which commutes with the Laplacian and $a_{ij}a^{ij} = \text{const}$, then M is the Riemannian product $M = M_0 \times M_1 \times \ldots \times M_k$ of simply connected totally geodesic submanifolds.*

2. Generalized curvature tensor fields.

2.1. DEFINITION [4]. Let M be a Riemannian manifold, $\dim M \geqslant 3$. A differentiable $(1,3)$-tensor L (with components $L^h{}_{ijk}$) will be called a *generalized curvature tensor on M* if

(2.1.1) $$L^h{}_{ijk} = -L^h{}_{ikj},$$

(2.1.2) $$g^{rh}L_{rijk} = g^{hr}L_{jkri},$$

(2.1.3) $$L^h{}_{ijk} + L^h{}_{jki} + L^h{}_{kij} = 0 \quad \text{(the first Bianchi identity)}.$$

We shall say that L is *proper* if L satisfies the second Bianchi identity

(2.1.4) $$\nabla_r L^h{}_{ijk} + \nabla_j L^h{}_{ikr} + \nabla_k L^h{}_{irj} = 0.$$

2.2. PROPOSITION ([4], p. 388, Proposition 2). *Let* $\dim M \geqslant 3$. *Let*

$$\Lambda(M) := \{L \mid L \textit{ generalized curvature tensor on } M\},$$

$$\Lambda_0(M) := \{L \in \Lambda(M) \mid L^{ik}{}_{ik} = 0 \textit{ on } M\},$$

$$\Lambda_1(M) := \{L \in \Lambda(M) \mid \langle L, L(0)\rangle := L_{hijk}L(0)^{hijk} = 0,\ L(0) \in \Lambda_0(M)\},$$

$$\Lambda_\omega(M) := \{L \in \Lambda_0(M) \mid L^h{}_{ijh} = 0\},$$

$$\Lambda_2(M) := \{L \in \Lambda_0(M) \mid \langle L, L(\omega)\rangle = 0,\ L(\omega) \in \Lambda_0(M)\}.$$

Then for every generalized curvature tensor field there is a natural direct sum decomposition

(2.2.1) $$L = L(1) + L(2) + L(\omega), \quad L(1) \in \Lambda_1(M), \quad L(2) \in \Lambda_2(M),$$

$L(\omega) \in \Lambda_\omega(M)$, *where*

(2.2.2) $$L(1)^h{}_{ijk} = \frac{1}{n(n-1)} R(L)(g_{ij}\delta^h_k - g_{ik}\delta^h_j),$$

(2.2.3) $$L(2)^h{}_{ijk} = \frac{1}{(n-2)}\big(R(L)_{ij}\delta^h_k - R(L)_{ik}\delta^h_j + g_{ij}R(L)^h{}_k - g_{ik}R(L)^h{}_j\big) + \frac{2}{n(n-2)} R(L)(g_{ik}\delta^h_j - g_{ij}\delta^h_k),$$

$$(2.2.4)\qquad L(\omega)^h{}_{ijk} = L^h{}_{ijk} + \frac{1}{(n-2)}\big(R(L)_{ik}\delta^h_j - R(L)_{ij}\delta^h_k + g_{ik}R(L)^h{}_j - g_{ij}R(L)^h{}_k\big) + \frac{1}{(n-1)(n-2)}R(L)(g_{ij}\delta^h_k - g_{ik}\delta^h_j);$$

$$(2.2.5)\qquad R(L)_{ij} := L^h{}_{ijh}$$

are the components of the Ricci tensor $\mathrm{Ric}(L)$,

$$(2.2.6)\qquad R(L) := R(L)^i{}_i$$

is the scalar curvature of L.

$L(\omega)$ is called the *conformal Weyl curvature tensor* of L.

The following lemma is obvious:

2.3. LEMMA. *Let* L *be a generalized curvature tensor on* M, $\dim M \geqslant 3$. *Then* $L(2) = 0$ *iff* $nR(L)_{ij} = R(L)g_{ij}$.

2.4. THEOREM. *Let* M *be a irreducible, simply connected Riemannian manifold,* $\dim M \geqslant 3$, *with nonpositive sectional curvature. Let* L *be a proper generalized curvature tensor over* M. *Then*

$$(2.4.1)\qquad \nabla_i R(L)_{jk} + \nabla_j R(L)_{ki} + \nabla_k R(L)_{ij} = 0 \quad \text{and} \quad R(L)^{ij}R(L)_{ij} = \mathrm{const}$$

imply

$$(2.4.2)\qquad L(2) = 0$$

on M.

Proof. (2.1.4) and (2.4.1) imply $R(L) = \mathrm{const}$; the assertion follows now from (1.9) and (2.3).

Proof of the Theorem. As M is homogeneous, we have $R = \mathrm{const}$ and $R^{ij}R_{ij} = \mathrm{const}$. From the commutativity of $\mathfrak{A}$ we have $D(\mathrm{Ric})\Delta = \Delta D(\mathrm{Ric})$, where $D(\mathrm{Ric}) = R^{ij}\nabla_i\nabla_j$. Then $\nabla(\mathrm{Ric}) = 0$ from (1.6).

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