

## ADAMS COMPLETION AND POSTNIKOV SYSTEMS

BY

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**1. Introduction.** Given a category  $\mathcal{D}$  and a family of morphisms  $S$  of  $\mathcal{D}$ , let  $\mathcal{D}[S^{-1}]$  be the category of fractions. For a given object  $Y$  of  $\mathcal{D}$ , we have a contravariant functor

$$\mathcal{D}[S^{-1}](-, Y): \mathcal{D} \rightarrow \text{Ens}.$$

If this functor is representable, then the representing object  $Y_S$  is called the (*generalized*) *Adams completion* of  $Y$  with respect to the family of morphisms  $S$  or the *S-completion* of  $Y$  (see [1]). This means that there is a natural equivalence of functors defined on  $\mathcal{D}$ :

$$\mathcal{D}[S^{-1}](-, Y) \cong \mathcal{D}(-, Y_S).$$

For the category  $\mathcal{D}$  of based CW-complexes and based maps and for the family of morphisms  $S$  rendered invertible by a homology theory defined on  $\mathcal{D}$ , Bousfield [2] and Deleanu [4] have shown that, for each object  $Y$  of  $\mathcal{D}$ , the corresponding Adams completion  $Y_S$  always exists. In [5], Deleanu has also shown that if certain conditions are imposed on an arbitrary category  $\mathcal{D}$  and a set of morphisms  $S$  of  $\mathcal{D}$ , then every object  $Y$  of  $\mathcal{D}$  has the Adams completion with respect to  $S$ .

We show in this paper that the notion of Adams completion and the Postnikov sections of a space  $X$  are intimately related. We formulate our results in the CW-category. For  $n \geq 1$ , let  $S_n$  be the family of  $(n+1)$ -equivalences in the CW-category. We show that, given a CW-complex  $X$ , the Adams completion of  $X$  with respect to  $S_n$  is precisely the Postnikov section of  $X$ . In what follows,  $\mathcal{C}$  denotes the category of 0-connected based CW-complexes and homotopy classes of maps. All maps and homotopies are assumed to be base-point preserving. By  $[X, Y]$  we mean the homotopy classes of base-point preserving maps from  $X$  to  $Y$ , and  $*$  will denote the base point for any space.

**2. The family of  $(n+1)$ -equivalences.** We now recall the following definitions.

**Definition 2.1.** A map  $f: X \rightarrow Y$  in  $\mathcal{C}$  is called an  $(n+1)$ -*equivalence* if  $f_*: \pi_m(X) \rightarrow \pi_m(Y)$  is an isomorphism for  $m \leq n$  and onto for  $m = n+1$ .

**Definition 2.2.** A family of morphisms  $S$  in a category  $\mathcal{D}$  is said to be *closed* if it contains the identities of  $\mathcal{C}$  and is closed under finite composition.

**Definition 2.3.** A family of morphisms  $S$  in a category  $\mathcal{D}$  is said to *admit a calculus of left fractions* (see [2]) if the following conditions are satisfied:

(i)  $S$  is closed under finite compositions and contains the identities of  $\mathcal{D}$ .

(ii) Any diagram

$$X_3 \xleftarrow{f} X_1 \xrightarrow{s} X_2 \quad \text{with } s \in S$$

can be embedded in a diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{s} & X_2 \\ f \downarrow & & \downarrow g \\ X_3 & \xrightarrow{t} & X_4 \end{array} \quad \text{with } t \in S.$$

(iii) Given

$$X_1 \xrightarrow{s} X_2 \xrightarrow[f]{g} X_3$$

in  $\mathcal{D}$  with  $s \in S$  and  $fs = gs$ , there exists  $X_3 \xrightarrow{t} X_4$  in  $\mathcal{D}$  such that  $tf = tg$ .

**PROPOSITION 2.1.** Let  $S$  be a closed family of morphisms of a category  $\mathcal{D}$  satisfying:

(i) if  $uv \in S$  and  $v \in S$ , then  $u \in S$ ;

(ii) every diagram

$$X_3 \xleftarrow{f} X_1 \xrightarrow{s} X_2$$

can be embedded in a weak push-out diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{s} & X_2 \\ f \downarrow & & \downarrow g \\ X_3 & \xrightarrow{t} & X_4 \end{array} \quad \text{with } t \in S.$$

Then  $S$  admits a calculus of left fractions.

For the proof see Theorem 1.3 of [6].

Let  $n \geq 1$  be a fixed integer and let  $S_n$  denote the set of all  $(n+1)$ -equivalences of  $\mathcal{C}$ . We prove the following

**PROPOSITION 2.2.**  $S_n$  admits a calculus of left fractions.

**Proof.** Clearly,  $S_n$  is closed. It is also clear that if  $uv \in S_n$  and  $v \in S_n$ , then  $u \in S_n$ ; thus condition (i) of Proposition 2.1 is satisfied. To prove that condition (ii) is also satisfied, let  $X_3 \xleftarrow{f} X_1 \xrightarrow{s} X_2$  be a given diagram in  $\mathcal{C}$  with  $s \in S_n$ . We replace  $X_2$  by the mapping cylinder  $M_s$  of the map  $s$ , and the map  $s$  by the usual inclusion  $i_s : X_1 \hookrightarrow M_s$ . We now form the push-out of  $i_s$  and  $f$  in  $\mathcal{C}$  (this is indeed a push-out in the category of all topological spaces):

$$\begin{array}{ccc} X_1 & \xrightarrow{i_s} & M_s \\ f \downarrow & & \downarrow \sigma \\ X_3 & \xrightarrow{t} & W \end{array}$$

Thus

$$W = X_3 \cup_f M_s.$$

Since  $s$  is an  $(n+1)$ -equivalence, so is  $i_s$  and, therefore,  $(M_s, X_1)$  is  $(n+1)$ -connected. It will now be enough to prove that  $t \in S_n$ .

We let

$$W_1 = \left( X_1 \times \left[ \frac{1}{2}, 1 \right] \right) \cup_s X_2 \quad \text{and} \quad W_2 = \left( X_1 \times \left[ 0, \frac{1}{2} \right] \right) \cup_f X_3.$$

Then

$$W = W_1 \cup W_2 \quad \text{and} \quad W_1 \cap W_2 = X_1 \times \left\{ \frac{1}{2} \right\} \cong X_1.$$

Moreover, it is evident that  $(W_1, X_1)$  is homotopically equivalent to  $(M_s, X_1)$  which is  $(n+1)$ -connected. We also infer that  $(W_2, X_1)$  is 0-connected and that  $X_1 \hookrightarrow W_1$  and  $X_1 \hookrightarrow W_2$  are cofibrations (since we are dealing with CW-complexes). Therefore, if  $j : (W_1, X_1, *) \rightarrow (W, W_2, *)$  denotes the usual inclusion, then

$$j_* : \pi_m(W_1, X_1, *) \rightarrow \pi_m(W, W_2, *)$$

is an  $(n+1)$ -isomorphism, i.e.,  $j_*$  is an isomorphism for  $m \leq n$  and onto for  $m = n+1$  (see [7], Theorem 16.29). Since  $(W_1, X_1, *)$  is  $(n+1)$ -connected, we have  $\pi_m(W_1, X_1, *) = 0$  for  $m \leq n+1$ , and hence  $\pi_m(W, W_2, *) = 0$  for  $m \leq n+1$ . Moreover, since  $W_2$  is homotopically equivalent to  $X_3$ , we have  $\pi_m(W, X_3, *) = 0$  for  $m \leq n+1$ . Thus  $t : X_3 \rightarrow W$  is an  $(n+1)$ -equivalence, and hence lies in  $S_n$ . This completes the proof of the proposition.

**3. Adams completion of a space with respect to  $S_n$ .** We shall show in this section that any object of  $\mathcal{C}$  has an Adams completion with respect to the family of morphisms  $S_n$ . We need the following theorem of Deleanu ([5], Theorem 1):

**THEOREM 3.1.** *Let  $\mathcal{U}$  be a fixed Grothendieck universe, let  $\mathcal{D}$  be a co-complete  $\mathcal{U}$ -category, and  $S$  a family of morphisms admitting a calculus of left fractions and satisfying the following axiom of comparability with co-products:*

(A) *If  $s_i: X_i \rightarrow Y_i$  lies in  $S$  for each  $i \in I$ , where the index set  $I$  is an element of the universe  $\mathcal{U}$ , then*

$$\coprod s_i: \coprod X_i \rightarrow \coprod Y_i$$

*lies in  $S$ .*

(B) *There exists a subset  $S_X$  of the set  $\{s: X \rightarrow X' \mid s \in S\}$  such that  $S_X$  is an element of the universe  $\mathcal{U}$  and for each  $s: X \rightarrow X'$ ,  $s \in S$ , there exist an  $s' \in S_X$  and a morphism  $u$  of  $\mathcal{D}$  rendering the following diagram commutative:*

$$\begin{array}{ccc}
 X & & \\
 \downarrow s & \searrow s' & \\
 X' & \xrightarrow{u} & X''
 \end{array}$$

*Then the Adams completion  $X_S$  of  $X$  does exist.*

In order to use this result, let  $\mathcal{U}$  be a fixed Grothendieck universe such that the category of OW-complexes and homotopy classes of maps between them is a  $\mathcal{U}$ -category. Since  $S^1$  can be given the structure of a OW-complex,  $[S^1, S^1] \simeq \mathbf{Z}$  is an element of  $\mathcal{U}$ , and it follows from the axioms of a Grothendieck universe that  $\mathbf{Z}^+$ , the set of positive integers, is also an element of  $\mathcal{U}$ . We shall use this fact in proving the following

**THEOREM 3.2.** *Every object  $X$  of  $\mathcal{C}$  has an Adams completion with respect to  $S_n$ .*

**Proof.** Let  $\mathcal{C}$  be a  $\mathcal{U}$ -category. Clearly,  $\mathcal{C}$  is cocomplete and  $S_n$  admits a calculus of left fractions by Proposition 2.2. It will now suffice to show that  $S_n$  satisfies conditions (A) and (B) of Theorem 3.1.

Let  $s_i: X_i \rightarrow Y_i$ ,  $i \in I$ , lie in  $S_n$ , i.e., each  $s_i$  is an  $(n+1)$ -equivalence. Then there are a relative OW-complex  $(Z_i, X_i)$  with cells in  $\dim \geq n+2$  and a map  $u_i: Z_i \rightarrow Y_i$  which is a weak homotopy equivalence (and, therefore, a homotopy equivalence) and such that  $u_i|_{X_i} = s_i$ . Therefore,

$$\left( \bigvee_{i \in I} Z_i, \bigvee_{i \in I} X_i \right)$$

is a relative CW-complex with cells in  $\dim \geq n+2$  and  $\bigvee_{i \in I} Z_i$  is homotopically equivalent to  $\bigvee_{i \in I} Y_i$ . It follows from these facts that

$$\pi_m(\bigvee_{i \in I} Z_i, \bigvee_{i \in I} X_i) = 0 \quad \text{for } m \leq n+1$$

showing that

$$\bigvee_{i \in I} s_i: \bigvee_{i \in I} X_i \rightarrow \bigvee_{i \in I} Y_i$$

is an  $(n+1)$ -equivalence; thus,  $S_n$  satisfies condition (A) of Theorem 3.1.

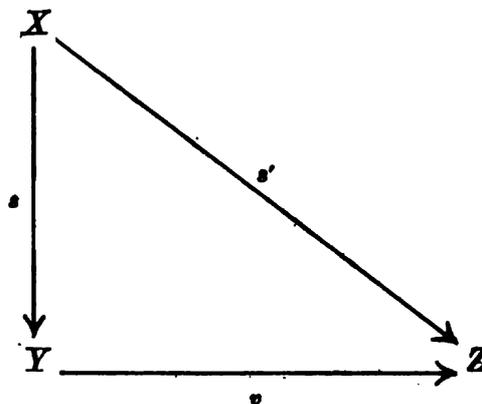
To prove that  $S_n$  satisfies condition (B) of Theorem 3.1, we proceed as follows. Given  $X$  in  $\mathcal{C}$ , we let

$$S_X = \{s: X \rightarrow Y \mid (Y, X) \text{ is a relative CW-complex with cells in } \dim \geq n+2\}.$$

Clearly,  $S_X \subset S_n$ . Moreover, if  $s: X \rightarrow Y$  is in  $S_n$ , then we can find a CW-complex such that

(i)  $(Z, X)$  has cells in  $\dim \geq n+2$ .

(ii) There is a map  $u: Z \rightarrow Y$  which is a homotopy equivalence and which extends  $s$ . If  $v$  denotes the homotopy inverse of  $u$  and  $s': X \rightarrow Z$  the usual inclusion, then the following diagram is easily seen to be homotopy commutative:



It will now be enough to prove that  $S_X \in \mathcal{U}$ . We write

$$A_k = \{s: X \rightarrow Y \mid (Y, X) \text{ has cells } e^m \text{ such that } n+2 \leq \dim(e^m) \leq n+k+1\},$$

so that we have  $S_X = \bigcup A_k$ ,  $k$  varying over the positive integers. We use

induction to show that, for every  $k \geq 1$ ,  $A_k \in \mathcal{U}$ . For  $k = 1$ , we have

$$A_1 = \{s: X \rightarrow Y \mid (Y, X) \text{ is a relative CW-complex} \\ \text{having cells in dim } n+2 \text{ only}\}.$$

Therefore,  $Y$  must be of the form

$$Y = X \cup_{a_i} e_i^{n+2},$$

where  $a_i: S^{n+1} \rightarrow X$  and  $i \in I$  for some index set  $I$ . It is also evident that every family  $\{a_i: S^{n+1} \rightarrow X\} \subset [S^{n+1}, X]$  determines a space  $Y$  such that  $(Y, X)$  is a relative CW-complex with cells in dim  $n+2$  only. Thus,  $A_1 \approx P[S^{n+1}, X]$ , where  $P$  denotes the power set. Since  $[S^{n+1}, X] \in \mathcal{U}$ , it follows from the axioms of a Grothendieck universe (see [3], p. 10) that  $P[S^{n+1}, X] \in \mathcal{U}$ ; thus  $A_1 \in \mathcal{U}$ .

We now assume inductively that  $A_k \in \mathcal{U}$ . To show that  $A_{k+1} \in \mathcal{U}$ , let  $s: X \rightarrow Y$  be in  $A_k$ , i.e.,  $(Y, X)$  is a relative CW-complex having cells  $e^m$  such that

$$n+2 \leq \dim(e^m) \leq n+k+1.$$

Let  $\{a_i\}_{i \in I}$  be a family of maps with  $a_i: S^{n+k+1} \rightarrow Y$  for some index set  $I$ . It is then clear that the inclusion

$$X \hookrightarrow Y \cup_{a_i} e_i^{n+k+2}$$

is in  $A_{k+1}$ . Moreover, every map  $s: X \rightarrow Z$  of  $A_{k+1}$  arises in this way. Therefore,

$$A_{k+1} = \bigcup_Y P[S^{n+k+1}, Y],$$

where the union is taken over all  $Y$  such that  $s: X \rightarrow Y$  is in  $A_k$ . Since  $A_k \in \mathcal{U}$  and  $P[S^{n+k}, Y] \in \mathcal{U}$ , we have  $A_{k+1} \in \mathcal{U}$ . Similarly, since the set of positive integers is an element of the universe  $\mathcal{U}$ , so is the union  $\bigcup A_k = S_X$ . This completes the proof of Theorem 3.2.

**4. Adams completion and Postnikov sections.** We show in this section that, for any space  $X$  in  $\mathcal{C}$ , the Adams completion  $X_n$  of  $X$  with respect to  $S_n$  is precisely the  $n$ -th Postnikov section of  $X$ . To prove this, we need a few results of a general nature.

**PROPOSITION 4.1.** *Let  $S$  be a family of morphisms of  $\mathcal{D}$ . If the object  $Z$  is the Adams completion of the object  $Y$ , then there is a map  $e: Y \rightarrow Z$  which is couniversal; i.e., given  $s: Y \rightarrow Z_1$  in  $S$ , there exists a unique morphism  $t: Z_1 \rightarrow Z$  such that  $ts = \theta$ .*

**Proof.** Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{D}[S^{-1}](Y, Y) & \xrightarrow[\cong]{\tau} & \mathcal{D}(Y, Z) \\ s^+ \uparrow & & \uparrow s^* \\ \mathcal{D}[S^{-1}](Z, Y) & \xrightarrow[\cong]{\tau} & \mathcal{D}(Z_1, Z) \end{array}$$

where  $s^+$  and  $s^*$  are defined as follows. Let  $F: \mathcal{D} \rightarrow \mathcal{D}[S^{-1}]$  be the canonical functor. Then, for any morphism  $\theta: Z_1 \rightarrow Y$  in  $\mathcal{D}[S^{-1}]$ , we define  $s^+(\theta) = \theta \circ F(s)$  and, for any morphism  $f: Z_1 \rightarrow Z$  in  $\mathcal{D}$ , we define  $s^*(f) = f \circ s$ . For  $s \in \mathcal{S}$ , the map  $s^+$  is a bijection since  $F(s)$  is an isomorphism; it is therefore evident that  $s^*$  is a bijection. We put  $e = \tau(1_Y)$ , where  $1_Y$  is the identity morphism of  $Y$  in  $\mathcal{D}[S^{-1}]$ . It is clear that there is a unique  $t$  in  $\mathcal{D}(Z_1, Z)$  such that  $s^*(t) = t \circ s = e$ .

**Remark.** It is not claimed that  $e \in \mathcal{S}$ . Similarly, the unique morphism  $t$  need not be in  $\mathcal{S}$ .

**PROPOSITION 4.2.** *Let  $Z$  be the Adams completion of  $Y$  and  $f: K \rightarrow Z$  a morphism in  $\mathcal{D}$ . Let  $g: K \rightarrow X$  be a morphism in  $\mathcal{D}$  such that  $F(g)$  is an isomorphism in  $\mathcal{D}[S^{-1}]$ . Then there exists a unique morphism  $h: X \rightarrow Z$  such that  $f = hg$ .*

**Proof.** The proof is only a slight variation of that of Proposition 4.1. We consider the commutative diagram

$$\begin{array}{ccc} \mathcal{D}[S^{-1}](K, Y) & \xrightarrow[\cong]{\tau} & \mathcal{D}(K, Z) \\ g^+ \uparrow & & \uparrow g^* \\ \mathcal{D}[S^{-1}](X, Y) & \xrightarrow[\cong]{\tau} & \mathcal{D}(X, Z) \end{array}$$

where  $g^+$  and  $g^*$  are defined as  $s^+$  and  $s^*$  in the proof of Proposition 4.1. Since  $F(g)$  is an isomorphism in  $\mathcal{D}[S^{-1}]$ ,  $g^+$  is bijective implying that  $g^*$  is bijective; so the required map  $h: X \rightarrow Z$  exists, and this completes the proof.

**PROPOSITION 4.3.** *Every object  $Y$  of  $\mathcal{D}$  admits an  $\mathcal{S}$ -completion if and only if the canonical functor  $F: \mathcal{D} \rightarrow \mathcal{D}[S^{-1}]$  has a right adjoint  $G$ . In that case, the  $\mathcal{S}$ -completion of  $Y$  is  $G(Y)$ .*

This is Corollary 2.2 of [6].

**PROPOSITION 4.4.** *If the canonical functor  $F: \mathcal{D} \rightarrow \mathcal{D}[S^{-1}]$  has a right adjoint  $G$ , then  $G$  is full and faithful.*

This is Proposition 2.3 of [6].

**PROPOSITION 4.5.** *If  $F: \mathcal{D} \rightarrow \mathcal{E}$ ,  $G: \mathcal{E} \rightarrow \mathcal{D}$ ,  $F \dashv G$ , and  $G$  is full and faithful, then the unit  $e: 1 \rightarrow GF$  of the adjunction belongs to the set of morphisms rendered invertible by  $F$ .*

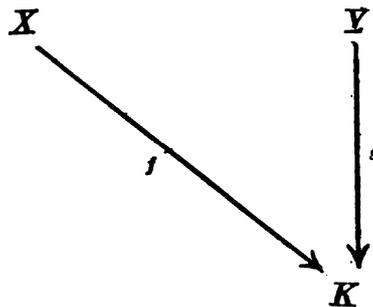
This is Proposition 2.4 of [6].

We now adapt these results to our situation, keeping  $n$  fixed for the remainder of this section. We have seen by Theorem 3.2 that every object of  $\mathcal{C}$  has an  $S_n$ -completion. Propositions 4.3 and 4.4 then imply that the canonical functor  $F: \mathcal{C} \rightarrow \mathcal{C}[S_n^{-1}]$  has a right adjoint  $G$  which is full and faithful. Moreover, for any object  $Y$  of  $\mathcal{C}$ , the Adams completion of  $Y$  is simply  $G(Y)$ . We can then apply Proposition 4.5 to conclude that the unit  $e: 1 \rightarrow GF$  of the adjunction belongs to the set of morphisms rendered invertible by  $F$ . Observe that, for any object  $Y$  of  $\mathcal{C}$ ,

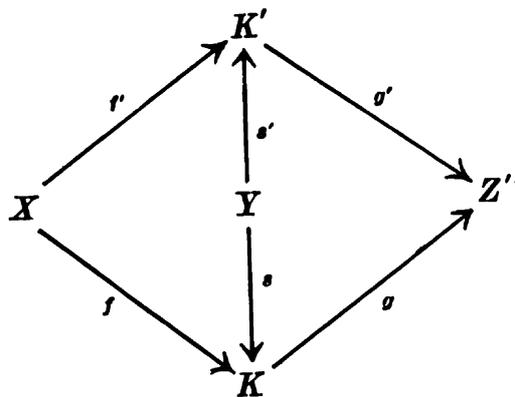
$$e(Y): Y \rightarrow GF(Y) = G(Y)$$

is the same map  $e: Y \rightarrow Z$  as constructed in Proposition 4.1.

We show next that this map  $e: Y \rightarrow Z$  (where  $Z$  is the  $S_n$ -completion of  $Y$ ) is in  $S_n$ . Recall that since  $S_n$  admits a calculus of left fractions, the objects and morphisms of the category  $\mathcal{C}[S_n^{-1}]$  can be explicitly described (see [5]). The objects of  $\mathcal{C}[S_n^{-1}]$  are the same as those of  $\mathcal{C}$ . A morphism  $\alpha: X \rightarrow Y$  in  $\mathcal{C}[S_n^{-1}]$  can be represented by a pair  $(f, s)$  with  $s \in S_n$ :

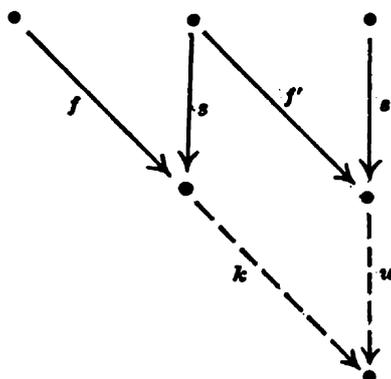


Any two such pairs  $(f, s)$  and  $(f', s')$  represent the same morphism  $\alpha$  if there exists a diagram

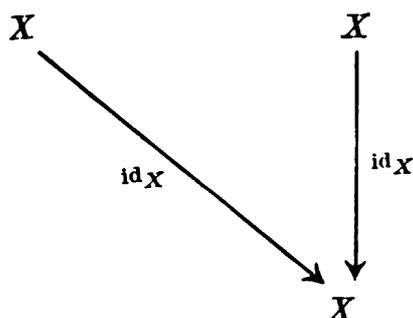


such that  $gs = g's' \in S_n$  and  $gf = g'f'$ . The composition of two morphisms represented by pairs  $(f, s)$  and  $(f', s')$  is represented by the pair  $(kf, us')$ , where  $u \in S_n$  and  $k$  are morphisms in  $\mathcal{C}$  such that the following

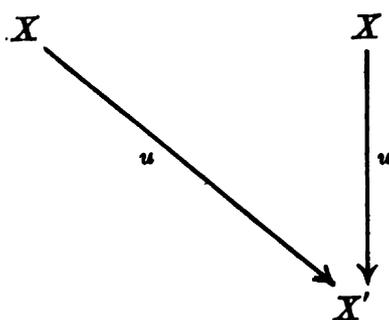
diagram is commutative:



Notice that the existence of the dotted arrows  $u$  and  $k$  is guaranteed by the fact that  $S_n$  admits a calculus of left fractions. The identity morphism  $1_X$  of  $X$  in  $\mathcal{C}[S_n^{-1}]$  is represented by a pair



(where  $\text{id}_X$  is the identity morphism of  $X$  in  $\mathcal{C}$ ) or, equivalently, by any pair



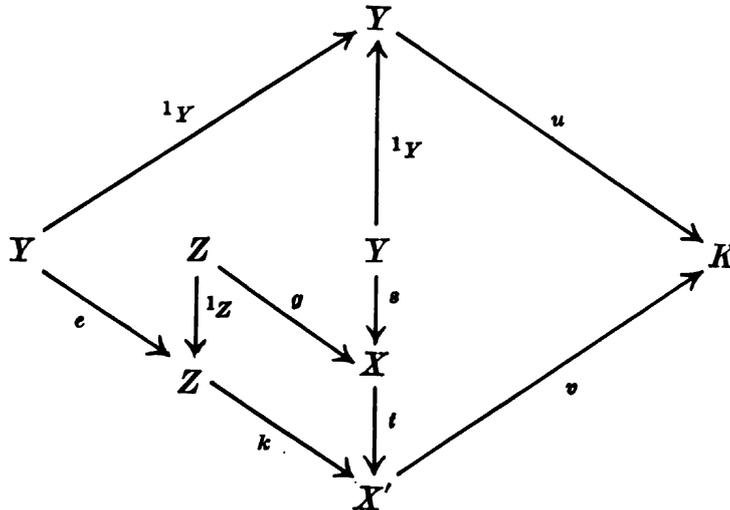
with  $u \in S_n$ . The canonical functor  $F: \mathcal{C} \rightarrow \mathcal{C}[S_n^{-1}]$  is then defined as follows. For an object  $X$  in  $\mathcal{C}$ ,  $F(X) = X$  and, for a morphism  $f: X \rightarrow Y$  in  $\mathcal{C}$ ,  $F(f) = a$ , where  $a$  is represented by the pair  $(f, 1_Y)$ .

**PROPOSITION 4.6.** *Let  $Y$  be an object of  $\mathcal{C}$  and  $Z$  its  $S_n$ -completion. Let  $e: Y \rightarrow Z$  be the map defined in Proposition 4.1. Then  $e$  is in  $S_n$ .*

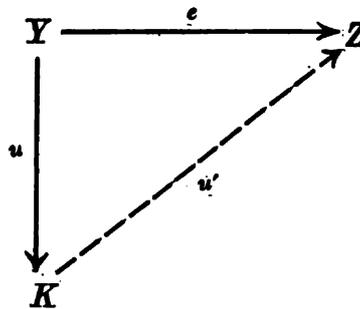
**Proof.** It follows from the discussion above that  $F(e)$  is an isomorphism in  $\mathcal{C}[S_n^{-1}]$ . Let  $(g, s)$  be the inverse of  $F(e) = (e, 1_Z)$ . Thus,

$$(g, s) \circ (e, 1_Z) = (1_Y, 1_Y).$$

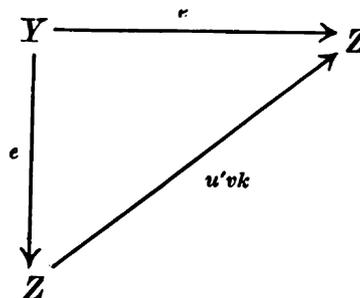
Therefore, we have a diagram



with  $u = vts \in S_n$  and  $u = vke$ . Since  $u_* = v_* k_* e_*$  and  $u_*$  is a monomorphism in  $\dim \leq n$ ,  $e_*$  is a monomorphism in  $\dim \leq n$ . Moreover, Proposition 4.1 implies that there is a unique map  $u': K \rightarrow Z$  with  $u'u = e$ :



We thus have  $e = u'u = u'vke$ :



Since  $F(e)$  is an isomorphism in  $\mathcal{C}[S_n^{-1}]$ , it follows from Proposition 4.2 that  $u'vk = 1_Z$ . Therefore,  $u'_* v_* k_* = {}^1\pi_*(Z)$ , which implies that  $u'_*$  is an epimorphism in all dimensions. This together with the fact that  $u_*$

is an epimorphism in  $\dim \leq n + 1$  implies that  $e_* = u'_*u_*$  is an epimorphism in  $\dim \leq n + 1$ . Thus,  $e \in S_n$ .

For each  $n$ , we denote this map from an object  $X$  to its  $S_n$ -completion  $X_n$  by  $e_n$ .

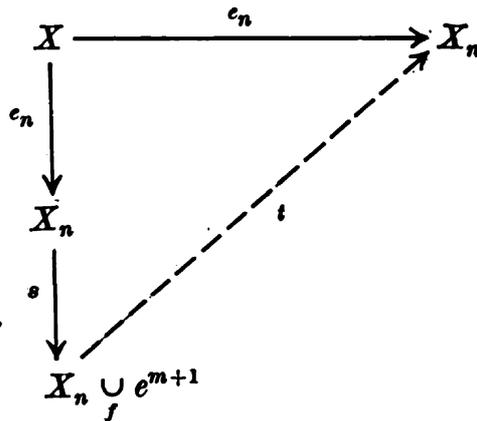
**THEOREM 4.1.** *For any  $X$  in  $\mathcal{C}$ , the  $S_n$ -completion  $X_n$  of  $X$  is the  $n$ -th Postnikov section of  $X$ .*

**Proof.** Since  $X_n$  is the  $S_n$ -completion of  $X$ , it follows from Proposition 4.6 that there is a map  $e_n: X \rightarrow X_n$  in  $S_n$  having the couniversal property. Since  $e_n \in S_n$ ,  $e_n$  is an  $(n + 1)$ -equivalence; thus  $\pi_m(X) \approx \pi_m(X_n)$  for  $m \leq n$ .

We now show that  $\pi_m(X_n) = 0$  for  $m > n$ . Let  $f: S^m \rightarrow X_n$  with  $m > n$ . Let  $s$  denote the inclusion

$$X_n \hookrightarrow X_n \underset{f}{\cup} e^{m+1};$$

then, clearly,  $s \in S_n$ . Consider the diagram



It follows from the couniversal property of  $e_n$  that there is a map

$$t: X_n \underset{f}{\cup} e^{m+1} \rightarrow X_n$$

such that  $ts\theta_n = \theta_n$ . This implies that  $f \simeq 0$ ; thus,  $\pi_m(X_n) = 0$  for  $m > n$ . Moreover, since  $e_n: X \rightarrow X_n$  is an  $(n + 1)$ -equivalence,  $(X_n, X)$  can be considered as a relative CW-complex with cells in  $\dim \geq n + 2$ .

Now we show that for every  $n \geq 1$  there exists a map  $\pi_{n+1}: X_{n+1} \rightarrow X_n$  such that  $\pi_{n+1}e_{n+1} = e_n$ . Indeed, since  $e_{n+1}: X \rightarrow X_{n+1}$  is an  $(n + 2)$ -equivalence, it is also an  $(n + 1)$ -equivalence, and the couniversal property of  $e_n$  implies that there is a map  $\pi_{n+1}: X_{n+1} \rightarrow X_n$  such that  $e_n = \pi_{n+1}e_{n+1}$ .

**Remarks.** (1) We follow the definition of Postnikov systems as given in [7]. The maps  $\{\pi_n\}$  can, of course, be replaced by fibrations in the usual manner.

(2) By applying the procedure given above, we get a  $K(\pi, n)$ -space as the  $S_n$ -completion of an  $M(\pi, n)$ -space.

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