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ADAMS COMPLETION AND POSTNIKOV SYSTEMS

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1. Introduction. Given a category $\mathscr{D}$ and a family of morphisms S of $\mathscr{D}$, let $\mathscr{D}[S^{-1}]$ be the category of fractions. For a given object Y of $\mathscr{D}$, we have a contravariant functor

$$\mathscr{D}[S^{-1}](-, Y)\colon \mathscr{D} \to \mathrm{Ens}.$$

If this functor is representable, then the representing object Y_S is called the (*generalized*) *Adams completion* of Y with respect to the family of morphisms S or the *S-completion* of Y (see [1]). This means that there is a natural equivalence of functors defined on $\mathscr{D}$:

$$\mathscr{D}[S^{-1}](-, Y) \cong \mathscr{D}(-, Y_S).$$

For the category $\mathscr{D}$ of based CW-complexes and based maps and for the family of morphisms S rendered invertible by a homology theory defined on $\mathscr{D}$, Bousfield [2] and Deleanu [4] have shown that, for each object Y of $\mathscr{D}$, the corresponding Adams completion Y_S always exists. In [5], Deleanu has also shown that if certain conditions are imposed on an arbitrary category $\mathscr{D}$ and a set of morphisms S of $\mathscr{D}$, then every object Y of $\mathscr{D}$ has the Adams completion with respect to S.

We show in this paper that the notion of Adams completion and the Postnikov sections of a space X are intimately related. We formulate our results in the CW-category. For $n \geqslant 1$, let S_n be the family of $(n+1)$-equivalences in the CW-category. We show that, given a CW-complex X, the Adams completion of X with respect to S_n is precisely the Postnikov section of X. In what follows, $\mathscr{C}$ denotes the category of 0-connected based CW-complexes and homotopy classes of maps. All maps and homotopies are assumed to be base-point preserving. By $[X, Y]$ we mean the homotopy classes of base-point preserving maps from X to Y, and $*$ will denote the base point for any space.

2. The family of $(n+1)$-equivalences. We now recall the following definitions.

Definition 2.1. A map $f\colon X \to Y$ in $\mathscr{C}$ is called an $(n+1)$-*equivalence* if $f_*\colon \pi_m(X) \to \pi_m(Y)$ is an isomorphism for $m \leqslant n$ and onto for $m = n+1$.

Definition 2.2. A family of morphisms S in a category $\mathscr{D}$ is said to be *closed* if it contains the identities of $\mathscr{C}$ and is closed under finite composition.

Definition 2.3. A family of morphisms S in a category $\mathscr{D}$ is said to *admit a calculus of left fractions* (see [2]) if the following conditions are satisfied:

(i) S is closed under finite compositions and contains the identities of $\mathscr{D}$.

(ii) Any diagram

$$X_3 \xleftarrow{f} X_1 \xrightarrow{s} X_2 \quad \text{with } s \in S$$

can be embedded in a diagram

$$\begin{tikzcd} X_1 \arrow[r, "s"] \arrow[d, "f"'] & X_2 \arrow[d, "g"] \\ X_3 \arrow[r, "t"'] & X_4 \end{tikzcd} \quad \text{with } t \in S.$$

(iii) Given

$$X_1 \xrightarrow{s} X_2 \overset{f}{\underset{g}{\rightrightarrows}} X_3$$

in $\mathscr{D}$ with $s \in S$ and $fs = gs$, there exists $X_3 \xrightarrow{t} X_4$ in $\mathscr{D}$ such that $tf = tg$.

PROPOSITION 2.1. *Let S be a closed family of morphisms of a category $\mathscr{D}$ satisfying*:

(i) *if $uv \in S$ and $v \in S$, then $u \in S$;*

(ii) *every diagram*

$$X_3 \xleftarrow{f} X_1 \xrightarrow{s} X_2$$

can be embedded in a weak push-out diagram

$$\begin{tikzcd} X_1 \arrow[r, "s"] \arrow[d, "f"'] & X_2 \arrow[d, "g"] \\ X_3 \arrow[r, "t"'] & X_4 \end{tikzcd} \quad \textit{with } t \in S.$$

Then S admits a calculus of left fractions.

For the proof see Theorem 1.3 of [6].

Let $n \geqslant 1$ be a fixed integer and let S_n denote the set of all $(n+1)$-equivalences of $\mathscr{C}$. We prove the following

PROPOSITION 2.2. *S_n admits a calculus of left fractions.*

Proof. Clearly, S_n is closed. It is also clear that if $uv \in S_n$ and $v \in S_n$, then $u \in S_n$; thus condition (i) of Proposition 2.1 is satisfied. To prove that condition (ii) is also satisfied, let $X_3 \xleftarrow{f} X_1 \xrightarrow{s} X_2$ be a given diagram in $\mathscr{C}$ with $s \in S_n$. We replace X_2 by the mapping cylinder M_s of the map s, and the map s by the usual inclusion $i_s : X \hookrightarrow M_s$. We now form the push-out of i_s and f in $\mathscr{C}$ (this is indeed a push-out in the category of all topological spaces):

$$\begin{tikzcd} X_1 \arrow[r, "i_s"] \arrow[d, "f"'] & M_s \arrow[d, "g"] \\ X_3 \arrow[r, "t"'] & W \end{tikzcd}$$

Thus

$$W = X_3 \cup_f M_s.$$

Since s is an $(n+1)$-equivalence, so is i_s and, therefore, (M_s, X_1) is $(n+1)$-connected. It will now be enough to prove that $t \in S_n$.

We let

$$W_1 = \left(X_1 \times \left[\frac{1}{2}, 1\right]\right) \cup_s X_2 \quad \text{and} \quad W_2 = \left(X_1 \times \left[0, \frac{1}{2}\right]\right) \cup_f X_3.$$

Then

$$W = W_1 \cup W_2 \quad \text{and} \quad W_1 \cap W_2 = X_1 \times \left\{\frac{1}{2}\right\} \cong X_1.$$

Moreover, it is evident that (W_1, X_1) is homotopically equivalent to (M_s, X_1) which is $(n+1)$-connected. We also infer that (W_2, X_1) is 0-connected and that $X_1 \hookrightarrow W_1$ and $X_1 \hookrightarrow W_2$ are cofibrations (since we are dealing with CW-complexes). Therefore, if $j\colon (W_1, X_1, *) \to (W, W_2, *)$ denotes the usual inclusion, then

$$j_*\colon \pi_m(W_1, X_1, *) \to \pi_m(W, W_2, *)$$

is an $(n+1)$-isomorphism, i.e., j_* is an isomorphism for $m \leqslant n$ and onto for $m = n+1$ (see [7], Theorem 16.29). Since $(W_1, X_1, *)$ is $(n+1)$-connected, we have $\pi_m(W_1, X_1, *) = 0$ for $m \leqslant n+1$, and hence $\pi_m(W, W_2, *) = 0$ for $m \leqslant n+1$. Moreover, since W_2 is homotopically equivalent to X_3, we have $\pi_m(W, X_3, *) = 0$ for $m \leqslant n+1$. Thus $t\colon X_3 \to W$ is an $(n+1)$-equivalence, and hence lies in S_n. This completes the proof of the proposition.

3. Adams completion of a space with respect to S_n. We shall show in this section that any object of $\mathscr{C}$ has an Adams completion with respect to the family of morphisms S_n. We need the following theorem of Deleanu ([5], Theorem 1):

THEOREM 3.1. *Let $\mathscr{U}$ be a fixed Grothendieck universe, let $\mathscr{D}$ be a cocomplete $\mathscr{U}$-category, and S a family of morphisms admitting a calculus of left fractions and satisfying the following axiom of comparability with coproducts:*

(A) *If $s_i\colon X_i \to Y_i$ lies in S for each $i \in I$, where the index set I is an element of the universe $\mathscr{U}$, then*

$$\coprod s_i\colon \coprod X_i \to \coprod Y_i$$

lies in S.

(B) *There exists a subset S_X of the set $\{s\colon X \to X' \mid s \in S\}$ such that S_X is an element of the universe $\mathscr{U}$ and for each $s\colon X \to X'$, $s \in S$, there exist an $s' \in S_X$ and a morphism u of $\mathscr{D}$ rendering the following diagram commutative:*

$$\begin{tikzcd} X \arrow[d, "s"'] \arrow[dr, "s'"] & \\ X' \arrow[r, "u"'] & X'' \end{tikzcd}$$

Then the Adams completion X_S of X does exist.

In order to use this result, let $\mathscr{U}$ be a fixed Grothendieck universe such that the category of CW-complexes and homotopy classes of maps between them is a $\mathscr{U}$-category. Since S^1 can be given the structure of a CW-complex, $[S^1, S^1] \simeq \mathbf{Z}$ is an element of $\mathscr{U}$, and it follows from the axioms of a Grothendieck universe that $\mathbf{Z}^+$, the set of positive integers, is also an element of $\mathscr{U}$. We shall use this fact in proving the following

THEOREM 3.2. *Every object X of $\mathscr{C}$ has an Adams completion with respect to S_n.*

Proof. Let $\mathscr{C}$ be a $\mathscr{U}$-category. Clearly, $\mathscr{C}$ is cocomplete and S_n admits a calculus of left fractions by Proposition 2.2. It will now suffice to show that S_n satisfies conditions (A) and (B) of Theorem 3.1.

Let $s_i\colon X_i \to Y_i$, $i \in I$, lie in S_n, i.e., each s_i is an $(n+1)$-equivalence. Then there are a relative CW-complex (Z_i, X_i) with cells in $\dim \geqslant n+2$ and a map $u_i\colon Z_i \to Y_i$ which is a weak homotopy equivalence (and, therefore, a homotopy equivalence) and such that $u_i | X_i = s_i$. Therefore,

$$\Big(\bigvee_{i\in I} Z_i, \bigvee_{i\in I} X_i\Big)$$

is a relative CW-complex with cells in $\dim \geqslant n+2$ and $\bigvee\limits_{i\in I} Z_i$ is homotopically equivalent to $\bigvee\limits_{i\in I} Y_i$. It follows from these facts that

$$\pi_m\big(\bigvee_{i\in I} Z_i, \bigvee_{i\in I} X_i\big) = 0 \quad \text{for } m \leqslant n+1$$

showing that

$$\bigvee_{i\in I} s_i \colon \bigvee_{i\in I} X_i \to \bigvee_{i\in I} Y_i$$

is an $(n+1)$-equivalence; thus, S_n satisfies condition (A) of Theorem 3.1.

To prove that S_n satisfies condition (B) of Theorem 3.1, we proceed as follows. Given X in $\mathscr{C}$, we let

$$S_X = \{s\colon X \to Y \mid (Y, X) \text{ is a relative CW-complex with cells in } \dim \geqslant n+2\}.$$

Clearly, $S_X \subset S_n$. Moreover, if $s\colon X \to Y$ is in S_n, then we can find a CW-complex such that

(i) (Z, X) has cells in $\dim \geqslant n+2$.

(ii) There is a map $u\colon Z \to Y$ which is a homotopy equivalence and which extends s. If v denotes the homotopy inverse of u and $s'\colon X \to Z$ the usual inclusion, then the following diagram is easily seen to be homotopy commutative:

$$\begin{tikzcd} X \arrow[d, "s"'] \arrow[dr, "s'"] & \\ Y \arrow[r, "v"'] & Z \end{tikzcd}$$

It will now be enough to prove that $S_X \in \mathscr{U}$. We write

$$A_k = \{s\colon X \to Y \mid (Y, X) \text{ has cells } e^m \text{ such that } n+2 \leqslant \dim(e^m) \leqslant n+k+1\},$$

so that we have $S_X = \bigcup A_k$, k varying over the positive integers. We use induction to show that, for every $k \geqslant 1$, $A_k \in \mathscr{U}$. For $k = 1$, we have

$$A_1 = \{s\colon X \to Y \mid (Y, X) \text{ is a relative CW-complex having cells in dim } n+2 \text{ only}\}.$$

Therefore, Y must be of the form

$$Y = X \underset{a_i}{\cup} e_i^{n+2},$$

where $a_i\colon S^{n+1} \to X$ and $i \in I$ for some index set I. It is also evident that every family $\{a_i\colon S^{n+1} \to X\} \subset [S^{n+1}, X]$ determines a space Y such that (Y, X) is a relative CW-complex with cells in dim $n+2$ only. Thus, $A_1 \approx P[S^{n+1}, X]$, where P denotes the power set. Since $[S^{n+1}, X] \in \mathscr{U}$, it follows from the axioms of a Grothendieck universe (see [3], p. 10) that $P[S^{n+1}, X] \in \mathscr{U}$; thus $A_1 \in \mathscr{U}$.

We now assume inductively that $A_k \in \mathscr{U}$. To show that $A_{k+1} \in \mathscr{U}$, let $s\colon X \to Y$ be in A_k, i.e., (Y, X) is a relative CW-complex having cells e^m such that

$$n+2 \leqslant \dim(e^m) \leqslant n+k+1.$$

Let $\{a_i\}_{i \in I}$ be a family of maps with $a_i\colon S^{n+k+1} \to Y$ for some index set I. It is then clear that the inclusion

$$X \hookrightarrow Y \underset{a_i}{\cup} e_i^{n+k+2}$$

is in A_{k+1}. Moreover, every map $s\colon X \to Z$ of A_{k+1} arises in this way. Therefore,

$$A_{k+1} = \bigcup_Y P[S^{n+k+1}, Y],$$

where the union is taken over all Y such that $s\colon X \to Y$ is in A_k. Since $A_k \in \mathscr{U}$ and $P[S^{n+k}, Y] \in \mathscr{U}$, we have $A_{k+1} \in \mathscr{U}$. Similarly, since the set of positive integers is an element of the universe $\mathscr{U}$, so is the union $\bigcup A_k = S_X$. This completes the proof of Theorem 3.2.

4. Adams completion and Postnikov sections. We show in this section that, for any space X in $\mathscr{C}$, the Adams completion X_n of X with respect to S_n is precisely the n-th Postnikov section of X. To prove this, we need a few results of a general nature.

PROPOSITION 4.1. *Let S be a family of morphisms of $\mathscr{D}$. If the object Z is the Adams completion of the object Y, then there is a map $e\colon Y \to Z$ which is couniversal; i.e., given $s\colon Y \to Z_1$ in S, there exists a unique morphism $t\colon Z_1 \to Z$ such that $ts = \theta$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathscr{D}[S^{-1}](Y, Y) \arrow[r, "\tau", "\simeq"'] & \mathscr{D}(Y, Z) \\ \mathscr{D}[S^{-1}](Z, Y) \arrow[u, "s^+"] \arrow[r, "\tau", "\simeq"'] & \mathscr{D}(Z_1, Z) \arrow[u, "s^*"'] \end{tikzcd}$$

where s^+ and s^* are defined as follows. Let $F\colon \mathscr{D} \to \mathscr{D}[S^{-1}]$ be the canonical functor. Then, for any morphism $\theta : Z_1 \to Y$ in $\mathscr{D}[S^{-1}]$, we define $s^+(\theta) = \theta \circ F(s)$ and, for any morphism $f\colon Z_1 \to Z$ in $\mathscr{D}$, we define $s^*(f) = f \circ s$. For $s \in S$, the map s^+ is a bijection since $F(s)$ is an isomorphism; it is therefore evident that s^* is a bijection. We put $e = \tau(1_Y)$, where 1_Y is the identity morphism of Y in $\mathscr{D}[S^{-1}]$. It is clear that there is a unique t in $\mathscr{D}(Z_1, Z)$ such that $s^*(t) = t \circ s = e$.

Remark. It is not claimed that $e \in S$. Similarly, the unique morphism t need not be in S.

PROPOSITION 4.2. *Let Z be the Adams completion of Y and $f\colon K \to Z$ a morphism in $\mathscr{D}$. Let $g\colon K \to X$ be a morphism in $\mathscr{D}$ such that $F(g)$ is an isomorphism in $\mathscr{D}[S^{-1}]$. Then there exists a unique morphism $h\colon X \to Z$ such that $f = hg$.*

Proof. The proof is only a slight variation of that of Proposition 4.1. We consider the commutative diagram

$$\begin{tikzcd} \mathscr{D}[S^{-1}](K, Y) \arrow[r, "\tau", "\simeq"'] & \mathscr{D}(K, Z) \\ \mathscr{D}[S^{-1}](X, Y) \arrow[u, "g^+"] \arrow[r, "\tau", "\simeq"'] & \mathscr{D}(X, Z) \arrow[u, "g^*"'] \end{tikzcd}$$

where g^+ and g^* are defined as s^+ and s^* in the proof of Proposition 4.1. Since $F(g)$ is an isomorphism in $\mathscr{D}[S^{-1}]$, g^+ is bijective implying that g^* is bijective; so the required map $h\colon X \to Z$ exists, and this completes the proof.

PROPOSITION 4.3. *Every object Y of $\mathscr{D}$ admits an S-completion if and only if the canonical functor $F\colon \mathscr{D} \to \mathscr{D}[S^{-1}]$ has a right adjoint G. In that case, the S-completion of Y is $G(Y)$.*

This is Corollary 2.2 of [6].

PROPOSITION 4.4. *If the canonical functor $F\colon \mathscr{D} \to \mathscr{D}[S^{-1}]$ has a right adjoint G, then G is full and faithful.*

This is Proposition 2.3 of [6].

PROPOSITION 4.5. *If $F\colon \mathscr{D} \to E$, $G\colon E \to \mathscr{D}$, $F \dashv G$, and G is full and faithful, then the unit $e\colon 1 \to GF$ of the adjunction belongs to the set of morphisms rendered invertible by F.*

This is Proposition 2.4 of [6].

We now adapt these results to our situation, keeping n fixed for the remainder of this section. We have seen by Theorem 3.2 that every object of $\mathscr{C}$ has an S_n-completion. Propositions 4.3 and 4.4 then imply that the canonical functor F: $\mathscr{C} \to \mathscr{C}[S_n^{-1}]$ has a right adjoint G which is full and faithful. Moreover, for any object Y of $\mathscr{C}$, the Adams completion of Y is simply $G(Y)$. We can then apply Proposition 4.5 to conclude that the unit e: $1 \to GF$ of the adjunction belongs to the set of morphisms rendered invertible by F. Observe that, for any object Y of $\mathscr{C}$,

$$e(Y)\colon\ Y \to GF(Y) = G(Y)$$

is the same map e: $Y \to Z$ as constructed in Proposition 4.1.

We show next that this map e: $Y \to Z$ (where Z is the S_n-completion of Y) is in S_n. Recall that since S_n admits a calculus of left fractions, the objects and morphisms of the category $\mathscr{C}[S_n^{-1}]$ can be explicitly described (see [5]). The objects of $\mathscr{C}[S_n^{-1}]$ are the same as those of $\mathscr{C}$. A morphism α: $X \to Y$ in $\mathscr{C}[S_n^{-1}]$ can be represented by a pair (f, s) with $s \in S_n$:

$$\begin{tikzcd}
X \arrow[dr, "f"'] & Y \arrow[d, "s"] \\
& K
\end{tikzcd}$$

Any two such pairs (f, s) and (f', s') represent the same morphism α if there exists a diagram

$$\begin{tikzcd}
& K' \arrow[dr, "g'"] & \\
X \arrow[ur, "f'"] \arrow[dr, "f"'] & Y \arrow[u, "s'"'] \arrow[d, "s"] & Z'' \\
& K \arrow[ur, "g"'] &
\end{tikzcd}$$

such that $gs = g's' \in S_n$ and $gf = g'f'$. The composition of two morphisms represented by pairs (f, s) and (f', s') is represented by the pair (kf, us'), where $u \in S_n$ and k are morphisms in $\mathscr{C}$ such that the following diagram is commutative:

$$\begin{tikzcd}
\bullet \arrow[dr, "f"'] & \bullet \arrow[d, "s"] \arrow[dr, "f'"] & \bullet \arrow[d, "s'"] \\
& \bullet \arrow[dr, dashed, "k"'] & \bullet \arrow[d, dashed, "u"] \\
& & \bullet
\end{tikzcd}$$

Notice that the existence of the dotted arrows u and k is guaranteed by the fact that S_n admits a calculus of left fractions. The identity morphism 1_X of X in $\mathscr{C}[S_n^{-1}]$ is represented by a pair

$$\begin{tikzcd}
X \arrow[dr, "\mathrm{id}_X"'] & X \arrow[d, "\mathrm{id}_X"] \\
& X
\end{tikzcd}$$

(where id_X is the identity morphism of X in $\mathscr{C}$) or, equivalently, by any pair

$$\begin{tikzcd}
X \arrow[dr, "u"'] & X \arrow[d, "u"] \\
& X'
\end{tikzcd}$$

with $u \in S_n$. The canonical functor F: $\mathscr{C} \to \mathscr{C}[S_n^{-1}]$ is then defined as follows. For an object X in $\mathscr{C}$, $F(X) = X$ and, for a morphism f: $X \to Y$ in $\mathscr{C}$, $F(f) = a$, where a is represented by the pair $(f, 1_Y)$.

PROPOSITION 4.6. *Let Y be an object of $\mathscr{C}$ and Z its S_n-completion. Let e: $Y \to Z$ be the map defined in Proposition* 4.1. *Then e is in S_n.*

Proof. It follows from the discussion above that $F(e)$ is an isomorphism in $\mathscr{C}[S_n^{-1}]$. Let (g, s) be the inverse of $F(e) = (e, 1_Z)$. Thus,

$$(g, s)\circ(e, 1_Z) = (1_Y, 1_Y).$$

Therefore, we have a diagram

$$\begin{tikzcd}
 & & Y \arrow[dr, "u"] & \\
Y \arrow[urr, "1_Y"] \arrow[dr, "e"'] & Z \arrow[d, "1_Z"] \arrow[dr, "g"] & Y \arrow[u, "1_Y"'] \arrow[d, "s"] & K \\
 & Z \arrow[dr, "k"'] & X \arrow[d, "t"] & \\
 & & X' \arrow[uur, "v"'] &
\end{tikzcd}$$

with $u = vts \in S_n$ and $u = vke$. Since $u_* = v_* k_* e_*$ and u_* is a monomorphism in $\dim \leqslant n$, e_* is a monomorphism in $\dim \leqslant n$. Moreover, Proposition 4.1 implies that there is a unique map $u'\colon K \to Z$ with $u'u = e$:

$$\begin{tikzcd}
Y \arrow[r, "e"] \arrow[d, "u"'] & Z \\
K \arrow[ur, dashed, "u'"'] &
\end{tikzcd}$$

We thus have $e = u'u = u'vke$:

$$\begin{tikzcd}
Y \arrow[r, "e"] \arrow[d, "e"'] & Z \\
Z \arrow[ur, "u'vk"'] &
\end{tikzcd}$$

Since $F(e)$ is an isomorphism in $\mathscr{C}[S_n^{-1}]$, it follows from Proposition 4.2 that $u'vk = 1_Z$. Therefore, $u'_* v_* k_* = 1_{\pi_*(Z)}$, which implies that u'_* is an epimorphism in all dimensions. This together with the fact that u_*

is an epimorphism in dim $\leqslant n+1$ implies that $e_* = u'_* u_*$ is an epimorphism in dim $\leqslant n+1$. Thus, $e \in S_n$.

For each n, we denote this map from an object X to its S_n-completion X_n by e_n.

THEOREM 4.1. *For any X in $\mathscr{C}$, the S_n-completion X_n of X is the n-th Postnikov section of X.*

Proof. Since X_n is the S_n-completion of X, it follows from Proposition 4.6 that there is a map $e_n\colon X \to X_n$ in S_n having the couniversal property. Since $e_n \in S_n$, e_n is an $(n+1)$-equivalence; thus $\pi_m(X) \approx \pi_m(X_n)$ for $m \leqslant n$.

We now show that $\pi_m(X_n) = 0$ for $m > n$. Let $f\colon S^m \to X_n$ with $m > n$. Let s denote the inclusion

$$X_n \hookrightarrow X_n \underset{f}{\cup} e^{m+1};$$

then, clearly, $s \in S_n$. Consider the diagram

$$\begin{tikzcd}
X \arrow[rr, "e_n"] \arrow[d, "e_n"'] & & X_n \\
X_n \arrow[d, "s"'] & & \\
X_n \underset{f}{\cup} e^{m+1} \arrow[uurr, dashed, "t"'] & &
\end{tikzcd}$$

It follows from the couniversal property of e_n that there is a map

$$t\colon X_n \underset{f}{\cup} e^{m+1} \to X_n$$

such that $ts\,\theta_n = \theta_n$. This implies that $f \simeq 0$; thus, $\pi_m(X_n) = 0$ for $m > n$. Moreover, since $e_n\colon X \to X_n$ is an $(n+1)$-equivalence, (X_n, X) can be considered as a relative CW-complex with cells in dim $\geqslant n+2$.

Now we show that for every $n \geqslant 1$ there exists a map $\pi_{n+1}\colon X_{n+1} \to X_n$ such that $\pi_{n+1} e_{n+1} = e_n$. Indeed, since $e_{n+1}\colon X \to X_{n+1}$ is an $(n+2)$-equivalence, it is also an $(n+1)$-equivalence, and the couniversal property of e_n implies that there is a map $\pi_{n+1}\colon X_{n+1} \to X_n$ such that $e_n = \pi_{n+1} e_{n+1}$.

Remarks. (1) We follow the definition of Postnikov systems as given in [7]. The maps $\{\pi_n\}$ can, of course, be replaced by fibrations in the usual manner.

(2) By applying the procedure given above, we get a $K(\pi, n)$-space as the S_n-completion of an $M(\pi, n)$-space.

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