COLLOQUIUM MATHEMATICUM

VOL. XXXIX 1978 FASC. 1

# *MARTINGALES IN BANACH SPACES FOR WHICH THE CONVERGENCE WITH PROBABILITY ONE, IN PROBABILITY AND IN LAW COINCIDE*

BY

A. KORZENIOWSKI (WROCŁAW)

**0. Introduction.** A general characterization of the convergence of sums of independent random variables in Banach spaces is due to Ito and Nisio [6]. It states that the convergence almost surely, in probability, and in law are equivalent. Deleting the independence assumption we find that even for martingales the situation changes. It is well known that a martingale convergent in probability need not be convergent almost surely [11] and for the convergence in probability the convergence in law is not sufficient either. Thus the question appears: under what assumptions on a martingale taking values in a Banach space the above convergences are equivalent? The answer contained in our Theorem 1 is, to the best of our knowledge, unknown even for the real line. As a corollary we also get a condition for a.s. convergence of an asymptotic martingale, the notion introduced and investigated by Chacon and Sucheston [3].

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \ldots \subset \mathscr{F}$ a sequence of sub-$\sigma$-fields. In order to avoid analytical problems connected with the measurable selection of random maps we assume the completeness of $P$. Let $E$ be a Banach space and $\mathscr{B}(E)$ the class of all Borel subsets. A random variable is a strongly measurable map from $\Omega$ into $E$. The integral is taken in the Bochner sense. The set of all bounded stopping times, i.e. taking a finite number of values ordered by $\leqslant$ a.s., is denoted by $T$. An adapted sequence $(X_n, \mathscr{F}_n)$ is said to be an *asymptotic martingale* (cf. [3]) if $(\int X_\sigma)_{\sigma\in T}$ converges, i.e. if there is a vector $z \in E$ with the property: for each $\varepsilon > 0$ there exists a $\sigma_0 \in T$ such that

$$\left\| \int X_\sigma - z \right\| < \varepsilon \quad \text{for } \sigma \geqslant \sigma_0.$$

If $\int X_\sigma$ is constant, then $X_n$ is said to be a *martingale.*

Throughout this paper, $E$ is assumed to be separable and $\tau$ stands for a locally convex Hausdorff topology on $E$ not stronger than the norm topology such that the unit ball of $E$ is $\tau$-closed. Recall that a sequence of probability measures $\mu_n$ on $(E, \mathscr{B}(E))$ *converges $\tau$-weakly* to a probability measure $\mu$ if

$$\int_E f(x)\mu_n(dx) \to \int_E f(x)\mu(dx)$$

for all $\tau$-continuous, bounded functions $f$ from $E$ into the real line. $\mu_n$ is said to be *$\tau$-conditionally compact* if its every subsequence $\mu_{n'}$ contains a subsequence $\mu_{n''}$ which converges $\tau$-weakly.

A sequence of random variables $X_n$ is said to be *$\tau$-relatively compact in probability* if its every subsequence $X_{n'}$ contains a subsequence $X_{n''}$ which converges in the topology $\tau$ in probability.

**1. Basic facts.**

PROPOSITION 1. *Every locally convex topology $\tau$ on $E$, not stronger than the norm topology, can be weakened to a locally convex topology $\tau_0$ with the following properties:*

(i) *$\tau_0$ is metrizable under a semi-norm $\|\cdot\|_0$,*

(ii) *$\tau_0$ generates all Borel sets $\mathscr{B}(E)$.*

Proof. Since $E$ is separable, there exists a countable weakly sequentially dense subset $C$ of $\tau$-continuous linear functionals. The set

$$D = \{x^* \in E \mid \|x^*\| \leqslant 1\} \cap C$$

generates a topology on $E$ (weaker than the weak topology of $E$) which is obviously locally convex and metrizable, e.g., by the semi-norm

$$\|x\|_0 = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x^*(x)|}{1+|x^*(x)|}, \qquad x \in E,\ x^* \in D.$$

It is clear that $\|\cdot\|_0$ satisfies (i) and (ii).

PROPOSITION 2. *Let $\mathscr{X}$ be a Polish space with the $\sigma$-field of Borel subsets $\mathscr{B}(\mathscr{X})$. Consider measurable maps*

$$\varphi\colon (\Omega, \mathscr{F}, P) \to (\mathscr{X}, \mathscr{B}(\mathscr{X})) \quad \text{and} \quad f\colon (E, \mathscr{B}(E)) \to (\mathscr{X}, \mathscr{B}(\mathscr{X}))$$

*with the property: $f^{-1}(x) \in \mathscr{B}(E)$ is closed for each $x \in \mathscr{X}$. If*

$$P(\{\omega \mid \varphi(\omega) \in f(E)\}) = 1,$$

*then there exists a random variable $X$ such that $\varphi$ can be factorized in the following way:*

$$\varphi(\omega) = f(X(\omega)) \text{ a.s.}$$

**Proof.** Define a multifunction $T\colon \mathscr{X} \to \mathscr{B}(E)$ as follows:

$$T(x) = \begin{cases} f^{-1}(x) & \text{if } x \in f(E), \\ 0 & \text{otherwise}. \end{cases}$$

Notice that $f(E)$, being an analytic set (cf. [9]), belongs to $\overline{\mathscr{B}}(\mathscr{X})$, where $\overline{\mathscr{B}}(\mathscr{X})$ is a $\varphi(P)$-completion of $\mathscr{B}(\mathscr{X})$ (cf. [10]). Since $T$ is closed-set valued, and the graph of $T$, i.e.

$$\{(x, y) \in \mathscr{X} \times E \mid x \in \mathscr{X}, y \in T(x)\},$$

is $\overline{\mathscr{B}}(\mathscr{X}) \times \mathscr{B}(E)$-measurable, so by Theorem 1 (cf. [12], p. 215-236) $T$ has the following property:

$$\{x \in \mathscr{X} \mid T(x) \cap G \neq \emptyset\} \in \overline{\mathscr{B}}(\mathscr{X}) \quad \text{for all open } G \in \mathscr{B}(E).$$

Now, by a general theorem on selectors (cf. [7]), there exists a single-valued measurable map

$$T_0\colon \big(\mathscr{X}, \overline{\mathscr{B}}(\mathscr{X})\big) \to \big(E, \mathscr{B}(E)\big)$$

such that $T_0(x) \in T(x)$ for all $x$. To complete the proof put $X(\omega) = T_0\big(\varphi(\omega)\big)$ and observe that, by the completeness of $P$, $X(\omega)$ is measurable as a superposition of measurable maps.

PROPOSITION 3. *Let $X_n$ be a sequence of $E$-valued random variables and let $\tau$ be a locally convex topology on $E$, not stronger than the norm topology. If $X_n(\omega)$ is $\tau$-relatively compact for almost all $\omega \in \Omega$, then there exists a random variable $X(\omega)$ such that $X(\omega)$ is a $\tau$-cluster point of $X_n(\omega)$ for almost all $\omega$.*

**Proof.** We need only to prove that $X(\omega)$ can be chosen in a measurable way. Let $C(\omega) = C\big(X_n(\omega)\big)$ denote the set of all $\tau$-cluster points of the sequence $X_n(\omega)$ for each $\omega$. Since the set of all $\tau_0$-cluster points of $\{X_n(\omega)\}$ is equal to $C(\omega)$ and since the graph of the $\tau_0$-closed set valued map $\omega \to C(\omega)$, i.e.

$$\{(\omega, y) \in \Omega \times E \mid \omega \in \Omega, y \in C(\omega)\} = \bigcap_{k=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{m=l}^{\infty} \left\{(\omega, y) \mid \|X_m(\omega) - y\|_0 < \frac{1}{k}\right\},$$

is by Proposition 1 $\mathscr{F} \times \mathscr{B}(E)$-measurable, by the same argument as in Proposition 2 we get a measurable selector $X(\omega)$.

PROPOSITION 4. *Let $X(\omega)$ be as in Proposition 3. Then there exists a sequence of strictly increasing stopping times $\sigma_n \in T$ such that $X_{\sigma_n}(\omega) \to X(\omega)$ in $\|\cdot\|_0$ a.s.*

The statement is known for the real line (cf. Lemma 1 in [1]) and in our case it can easily be obtained by using the real-line result. We omit the proof.

In [3] we find the following:

FACT 1. *If* W *is a property of an asymptotic martingale* $(X_n)$, *then*

$$\left(\mathrm{W} \wedge \sup_T \int \|X_\sigma\| < \infty\right) \Rightarrow X_n \text{ converges a.s.}$$

*whenever*

$$\left(\mathrm{W} \wedge \sup_T \int \|X_\sigma\| < \infty \wedge \sup_n \|X_n\| \in L_1\right) \Rightarrow X_n \text{ converges a.s.}$$

FACT 2. *For an asymptotic martingale with* $\sup_n \|X_n\| \in L_1$, *the limit*

$$\lim_T \int_A X_\sigma$$

*exists for any* $A$ *from the* $\sigma$*-field generated by* $\bigcup \mathscr{F}_n$.

Note that, in the martingale case, $L_1$-boundedness, i.e.

$$\sup_n \int \|X_n\| < \infty,$$

implies

$$\sup_T \int \|X_\sigma\| < \infty,$$

whence both facts follow immediately.

**2. Main result.**

THEOREM 1. *Let* $X_n$ *be an* $E$*-valued martingale with the corresponding sequence of probability laws* $\mu_n$ *and let* $\tau$ *be a locally convex topology on* $E$, *not stronger than the norm topology. If*

$$\sup_n \int \|X_n\| < \infty, \tag{1}$$

*then the following conditions are equivalent:*

(a) $X_n$ *converges a.s.*,

(b) $X_n$ *converges in probability*,

(c) $X_n(\omega)$ *is* $\tau$*-relatively compact for almost all* $\omega$,

(d) $X_n$ *is* $\tau$*-relatively compact in probability*,

(e) $\mu_n$ *converges in law*,

(f) $\mu_n$ *is* $\tau$*-conditionally compact.*

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c), (a) $\Rightarrow$ (e) $\Rightarrow$ (f), and (a) $\Rightarrow$ (d) are evident. Therefore, it remains to prove (c) $\Rightarrow$ (a), (d) $\Rightarrow$ (a), and (f) $\Rightarrow$ (a).

(f) $\Rightarrow$ (a). Let $D$ be as in the proof of Proposition 1. By (1), for $x^* \in D$, $x^*(X_n)$, being a real-valued $L_1$-bounded martinagle, converges to $l(x^*)$ outside a $P$-null set $N(D)$. We prove that $X_n$ converges in $\|\cdot\|_0$, i.e. there exists a random variable $X(\omega)$ such that

$$x^*(X_n) \to x^*(X) \text{ a.s.}, \qquad x^* \in D.$$

By the assumption, take $\mu_{n_k}$ which converges $\tau$-weakly to $\mu$. For real $t_1, t_2, \ldots, t_m$ and $x_1^*, x_2^*, \ldots, x_m^* \in D$ we have

$$\int_\Omega \exp\Big[i \sum_{j=1}^m t_j l(x_j^*)\Big] P(d\omega) = \lim_{n_k} \int_\Omega \exp\Big[i \sum_{j=1}^m t_j x_j^*(X_{n_k})\Big] P(d\omega)$$

$$= \lim_{n_k} \int_E \exp\Big[i\Big(\sum_{j=1}^m t_j x_j^*\Big)(x)\Big] \mu_{n_k}(dx) = \int_E \exp\Big[i \sum_{j=1}^m t_j x_j^*(x)\Big] \mu(dx).$$

Therefore, $R^\infty$-valued maps

$$\varphi(\omega) = \big(l(x^*)\big)_{x^* \in D} \quad \text{and} \quad f(x) = \big(x^*(x)\big)_{x^* \in D}$$

defined on the probability spaces $(\Omega, \mathscr{F}, P)$ and $\big(E, \mathscr{B}(E), \mu\big)$, respectively, have the same distribution law, whence

$$P\big(\omega \mid \varphi(\omega) \in f(E)\big) = 1.$$

Now from Proposition 2 for $\mathscr{X} = \big(R^\infty, \mathscr{B}(R^\infty)\big)$, and for $\varphi$ and $f$ as above we get the factorization

$$l(x^*) = x^*\big(X(\omega)\big) \text{ a.s.}, \qquad x^* \in D,$$

i.e. $x^*(X_n) \to x^*(X)$.

To prove the a.s. strong convergence consider a countable family of real-valued martingales $x^*(X_n)$, $x^* \in D$. From (1) and the Hahn-Banach theorem for $D$ ($\sup_{x^* \in D} x^*(x) = \|x\|$) it follows that

$$\sup_n \int \sup_{x^* \in D} x^*(X_n) = \sup_n \int \|X_n\| < \infty.$$

Therefore, from Lemma V-2-9 in [8] we obtain

$$\sup_{x^* \in D} x^*(X_n) \to \sup_{x^* \in D} x^*(X) \text{ a.s.} \tag{2}$$

For each $a \in E$ we have

$$\|X_n(\omega) - a\| = \sup_{x^* \in D} x^*\big(X_n(\omega) - a\big)$$

outside a $P$-null set $N(a, D)$. By (2) we get

$$\lim_n \|X_n(\omega) - a\| = \|X(\omega) - a\|$$

outside the $P$-null set $N(a, D)$, where $a$ belongs to any countable dense set in $E$. Thus we obtain

$$\lim_n \|X_n(\omega) - X(\omega)\| = 0 \text{ a.s.}$$

(d) $\Rightarrow$ (a). From the assumption and Proposition 1 it follows that there is a subsequence $X_{n_k}$ which converges a.s. to $X$ in $\|\cdot\|_0$, which implies the strong convergence by the first part of the proof.

(c) $\Rightarrow$ (a) It suffices to show that $X_n$ converges in $\|\cdot\|_0$. Assume the contrary. By Proposition 3 we may choose random variables $X$ and $X'$

such that $X \neq X'$ on a $P$-positive set $A$ belonging to the $\sigma$-field spanned by $\bigcup \mathscr{F}_n$. Moreover, $A$ can be taken in such a way that the corresponding integrals over $A$ are also different. Next, by Proposition 4, there are sequences $\sigma_n$ and $\sigma_{n'}$ of stopping times for which

$$\|X - X_{\sigma_n}\|_0 \to 0, \quad \|X' - X_{\sigma_{n'}}\|_0 \to 0 \quad \text{a.s.}$$

Let $x_0^* \in D$ separate $\int_A X$ and $\int_A X'$. From Fact 1 and from the Lebesgue dominated convergence theorem it follows that

$$\lim_n \int_A x_0^*(X_{\sigma_n}) = \int_A x_0^*(X) \neq \int_A x_0^*(X') = \lim_n \int_A x_0^*(X_{\sigma_{n'}}),$$

whence $\int_A x_0^*(X_\sigma)$ diverges, which contradicts Fact 2.

Theorem 1 gives a condition on a.s. convergence for martingales in Banach spaces without RNP (Radon-Nikodym property). In a space with RNP, $L_1$-boundedness suffices for a.s. convergence. This follows immediately from our theorem. In fact, by virtue of Fact 1, $\lim_n \int_A X_n$, where $A \in \sigma\left(\bigcup \mathscr{F}_n\right)$, is a vector measure with bounded total variation, so it has an integral representation by a random variable $X$. Therefore,

$$\lim_n \int_A X_n = \int_A X, \quad A \in \sigma\left(\bigcup \mathscr{F}_n\right),$$

so (c) is trivially fulfilled, since $X_n$ converges in $\|\cdot\|_0$.

THEOREM 2. *Let $X_n$ be an $E$-valued asymptotic martingale such that*

$$\sup_T \int \|X_\sigma\| < \infty \tag{3}$$

*and let $\tau$ be a locally convex topology on $E$, not stronger than the norm topology. Then the following conditions are equivalent:*

(a) *$X_n$ converges $\tau$ a.s.,*

(b) *$X_n(\omega)$ is $\tau$-relatively compact for almost all $\omega \in \Omega$.*

For the proof see (c) $\Rightarrow$ (a) in the proof of Theorem 1.

COROLLARY. *If $E$ is $\tau$-compactly generated (i.e. a unit ball of $E$ is $\tau$-compact as, for instance, in all duals with $\tau$ being a weak *-topology), then an asymptotic martingale satisfying* (3) *converges $\tau$ a.s.*

Proof. By the maximal Lemma 1 of [3] we have

$$P(\sup_n \|X_n\| > k) < \frac{1}{k} \sup_T \int \|X_\sigma\|,$$

thus (b) of Theorem 2 holds.

Acknowledgement. I am indebted to Professor Wojbor A. Woyczyński for his interest in and discussions on this paper.

## REFERENCES


[1] D. G. Austin, G. A. Edgar and A. Ionescu Tulcea, *Pointwise convergence in terms of expectations*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 30 (1974), p. 17-26.

[2] J. R. Baxter, *Pointwise in terms of weak convergence*, Proceedings of the American Mathematical Society 46 (1974), p. 395-398.

[3] R. V. Chacon and L. Sucheston, *On convergence of vector-valued asymptotic martingales*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 33 (1975), p. 55-59.

[4] S. D. Chatterji, *Martingale convergence and the Radon Nikodym theorem in Banach spaces*, Mathematica Scandinavica 22 (1968), p. 21-41.

[5] J. Hoffman-Jørgensen, *Sums of independent Banach space valued random variables*, Studia Mathematica 52 (1974), p. 159-186.

[6] K. Ito and M. Nisio, *On the convergence of sums of independent Banach space valued random variables*, Osaka Mathematical Journal 5 (1968), p. 35-48.

[7] K. Kuratowski and C. Ryll-Nardzewski, *A general theorem on selectors*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 13 (1965), p. 397-404.

[8] J. Neveu, *Martingales à temps discret*, Paris 1972.

[9] K. R. Parthasarathy, *Probability measures on metric spaces*, London 1967.

[10] M. Sion, *Topological and measure theoretic properties of analytic sets*, Proceedings of the American Mathematical Society 11 (1960), p. 769-776.

[11] W. F. Stout, *Almost sure convergence*, New York 1974.

[12] R. Tyrrel Rockafellar, *Convex integral functionals and duality* in *Contributions to nonlinear functional analysis*, New York - London 1971.


INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY

*Reçu par la Rédaction le 1. 6. 1976*