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ON SOME NUMERICAL REPRESENTATION OF POST ALGEBRAS

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1. Introduction. It was shown by Mączyński in [4] that every Boolean algebra can be isomorphically represented as a numerical Boolean algebra, which is, roughly speaking, a set of mappings from a non-empty set X into the closed interval $[0,1]$, with the Boolean structure obtained by means of the usual ordering of functions such that some Boolean fundamental operations are represented by numerical ones: the complementation is represented by the subtraction from 1, and the union of disjoint elements by the arithmetical sum.

It was shown in [5] that a similar representation is possible in the more general case of orthomodular partially ordered sets admitting a full set of measures. This representation was used by the first author in proving the theorem in [3].

In Section 3 of the present paper the definition of a numerical Post algebra is given, and some necessary and sufficient conditions are stated and proved for a partially ordered set of functions to be a Post algebra with respect to the usual ordering of functions. In Section 4 it is shown that every Post algebra P has a numerical representation. Since a numerical representation may be built up from every full set of measures on a given Post algebra P, in Section 5 we discuss the question of when and how a full set of measures on P arises from a full set of measures on the Boolean center of P.

2. Preliminary definitions and notation. Let $(P; e_0, \ldots, e_{n-1})$ denote a Post algebra of order n. It is well known that P is a bounded distributive lattice and $0 = e_0 < \ldots < e_{n-1} = 1$ is an n-term chain in P such that each member x of P has a representation of the form

$$x = (x_1 \cap e_1) \cup \ldots \cup x_{n-1} = \bigcup_{i=1}^{n-1} (x_i \cap e_i), \tag{2.1}$$

where x_i is a complemented element of P. If $x_i \cap x_j = 0$ for $i \neq j$, representation (2.1) is called *disjoint*, and in this case the coefficients x_i are usually denoted by $C_i(x)$, $i = 1, \ldots, n-1$. Each member $x \in P$ has exactly one disjoint representation. The center of all complemented elements of P will be denoted by B, and x' will stand for the complement of $x \in B$.

Let m_B be a *normed measure* on B, i.e. m_B is a mapping from B into the closed interval $[0, 1]$ such that $m_B(1) = 1$ and $m_B(a \cup b) = m_B(a) + m_B(b)$ provided that $a \cap b = 0$. If $a_1, \ldots, a_{n-1}$ are any numbers such that $0 = a_0 < a_1 < \ldots < a_{n-1} = 1$, then the function

(2.2)
$$m(x) = m_B\big(C_1(x)\big)a_1 + \ldots + m_B\big(C_{n-1}(x)\big)a_{n-1} = \sum_{i=1}^{n-1} m_B\big(C_i(x)\big)a_i, \qquad x \in P,$$

is a measure on the Post algebra P (see [6]). If m_B is a two-valued measure on B, then the measure m takes on only n values $0, a_1, \ldots, a_{n-1}$ and is called *n-valued*.

The set of all measures m defined by (2.2), where m_B runs over any set M_B of normed measures m_B on B, will be denoted by $(M_B; a_0, \ldots, a_{n-1})$. The set $M = (M_B; a_0, \ldots, a_{n-1})$ is said to be *full* if $m(x) \leqslant m(y)$ for all $m \in M$ implies $x \leqslant y$.

We say that M_B *induces a full set of measures* on P provided that there exists a chain of numbers $0 = a_0 < a_1 < \ldots < a_{n-1} = 1$ such that the set $M = (M_B; a_0, \ldots, a_{n-1})$ is full. If M_B is the set of all two-valued measures on B, then the set M is full for every fixed chain of constants $a_0, \ldots, a_{n-1}$ of the above kind.

In fact, if $x \nleqslant y$, then there exists a j such that

$$\bigcup_{i=j}^{n-1} C_i(x) \nleqslant \bigcup_{i=j}^{n-1} C_i(y).$$

Since M_B is full, there exists a measure $m_0 \in M_B$ such that

$$m_0\Big(\bigcup_{i=j}^{n-1} C_i(y)\Big) = 0 \quad \text{and} \quad m_0\Big(\bigcup_{i=j}^{n-1} C_i(x)\Big) = 1,$$

i.e. $m_0\big(C_i(x)\big) = 1$ for a certain $i \geqslant j$. Hence

$$m(y) = \sum_{i=1}^{n-1} m_0\big(C_i(y)\big)a_i = \sum_{i=1}^{j-1} m_0\big(C_i(y)\big)a_i \leqslant a_{j-1} \quad \text{and} \quad m(x) \geqslant a_i \geqslant a_j.$$

Since $a_{j-1} < a_j$, we conclude that $m(x) > m(y)$. So we proved that every Post algebra admits a full set of measures.

Now, let $[0, 1]^X$ denote the set of all functions from a non-empty set X into the closed interval $[0, 1]$. For all functions a, b in $[0, 1]^X$, $a+b$ and $a-b$ denote the sum and the difference of a and b, respectively; $a \leqslant b$ means that $a(x) \leqslant b(x)$ for all $x \in X$. The least and the greatest functions in $[0, 1]^X$ will be denoted by 0 and 1, respectively. The partially ordered set $([0, 1]^X, \leqslant)$ is a complete distributive lattice.

A subsystem $(A, \leqslant)$ of $([0,1]^X, \leqslant)$ is said to be a *numerical Boolean algebra* if it is a Boolean algebra with respect to $\leqslant$ and to the relative complementation $g \cap f' = g - f$ for $f \leqslant g$, $f, g \in A$. Mączyński proved in [4] that every Boolean algebra can be isomorphically represented as a numerical Boolean algebra, and that a subsystem $(A, \leqslant)$ of $([0,1]^X, \leqslant)$ is a numerical Boolean algebra if and only if the following conditions are satisfied:

1° The zero function 0 belongs to A.

2° For every $a \in A$, $1 - a$ belongs to A.

3° For every triple $a_1, a_2, a_3 \in A$, $a_i + a_j \leqslant 1$ for $i \neq j$ implies $a_1 + a_2 + a_3 \in A$.

4° For every pair $a, b \in A$, there are $c_1, c_2, c_3 \in A$ such that $c_i + c_j \leqslant 1$ for $i \neq j$, $a = c_1 + c_2$, and $b = c_2 + c_3$.

Notice that $a \leqslant b$ in a numerical Boolean algebra A implies $b - a \in A$.

3. Numerical Post algebras. Let $(P, \leqslant)$ be a subsystem of $([0,1]^X, \leqslant)$, where $\leqslant$ is the natural function ordering. We say that P is a *numerical Post algebra* if the following conditions are satisfied:

5° P is a distributive lattice with respect to $\leqslant$.

6° The center B of P is a numerical Boolean algebra.

7° There exists a chain of constant functions $0 = a_0 < a_1 < \ldots < a_{n-1} = 1$ in P such that if $f_1 + \ldots + f_{n-1} \leqslant 1$ and $f_i \in B$ for $i = 1, \ldots, n-1$, then

$$(f_1 \cap a_1) \cup \ldots \cup (f_{n-1} \cap a_{n-1}) = f_1 a_1 + \ldots + f_{n-1}$$

with numerical operations on the right-hand side of this formula.

8° For every $x \in P$, there exist $f_i \in B$ $(i = 1, \ldots, n-1)$ such that

$$x = \sum_{i=1}^{n-1} f_i a_i \quad \text{and} \quad f_1 + \ldots + f_{n-1} \leqslant 1, \tag{3.1}$$

and this representation is unique.

It is clear that $(P; a_0, \ldots, a_{n-1})$ is a Post algebra with respect to the natural order of functions.

Representation (3.1) is disjoint, since $f_1 + \ldots + f_{n-1} \leqslant 1$ implies $f_i \cap f_j = 0$ for $i \neq j$ (see [4]).

THEOREM 1. *Let M be a non-empty set and let P be a set of functions from M into* $[0, 1]$. *Assume that the following conditions are satisfied:*

I. *There exists a subset $B \subset P$ which is a numerical Boolean algebra with respect to the natural ordering of functions.*

II. *There exists a chain of constant functions $0 = a_0 < a_1 < \ldots < a_{n-1} = 1$ in P such that*

(a) $x = \sum_{i=1}^{n-1} f_i a_i \in P$ *whenever* $\sum_{i=1}^{n-1} f_i \leqslant 1$, $f_i \in B$ *for all* $i = 1, \ldots, n-1$;

(b) *every* $x \in P$ *has an* (a)-*type representation*;

(c) *for all* $x, y \in P$,

$$x \leqslant y \quad \text{if and only if} \quad \sum_{i=k}^{n-1} f_i \leqslant \sum_{i=k}^{n-1} g_i \quad \text{for } k = 1, \ldots, n-1$$

provided $x = f_1 a_1 + \ldots + f_{n-1}$ *and* $y = g_1 a_1 + \ldots + g_{n-1}$ *are* (a)-*type representations of* x *and* y, *respectively.*

Then $(P, \leqslant)$ *is a numerical Post algebra with respect to the natural ordering of functions.*

Proof. First of all we show that an (a)-type representation of an $x \in P$ is unique. In fact, if

$$x = f_1 a_1 + \ldots + f_{n-1} = g_1 a_1 + \ldots + g_{n-1}$$

are two representations of that type, then $f_{n-1} = g_{n-1}$ follows immediately from (c). Assume that $f_i = g_i$ for $i > k$. Since, by (c),

$$\sum_{i=k}^{n-1} f_i = \sum_{i=k}^{n-1} g_i,$$

we have

$$f_k = \sum_{i=k}^{n-1} f_i - \sum_{i=k+1}^{n-1} f_i = \sum_{i=k}^{n-1} g_i - \sum_{i=k+1}^{n-1} g_i = g_k.$$

So, by inductive argument, we obtain $f_i = g_i$ for all $i = 1, \ldots, n-1$.

In the second step of the proof we show that $(P, \leqslant)$ is a distributive lattice. Let us consider $x, y \in P$ with the (a)-type representations

$$x = \sum_{i=1}^{n-1} f_i a_i \quad \text{and} \quad y = \sum_{i=1}^{n-1} g_i a_i.$$

Let, by definition,

$$h_i = \Big(\sum_{j=i}^{n-1} f_j \cup \sum_{j=i}^{n-1} g_j\Big) - \Big(\sum_{j=i+1}^{n-1} f_j \cup \sum_{j=i+1}^{n-1} g_j\Big) \quad \text{for } i = 1, \ldots, n-2 \tag{3.2}$$

and

$$h_{n-1} = f_{n-1} \cup g_{n-1},$$

where the least upper bounds are taken in the Boolean algebra B. Evidently, $h_i \in B$ for $i = 1, \ldots, n-1$ (see the last observation in Section 2), and

$$h_1+\ldots+h_{n-1} = \sum_{i=1}^{n-1} f_i \cup \sum_{i=1}^{n-1} g_i \leqslant 1.$$

Hence

(3.3) $$z = h_1 a_1 + \ldots + h_{n-1}$$

is in P, according to (a). Since

$$\sum_{i=j}^{n-1} f_i \leqslant \sum_{i=j}^{n-1} h_i \quad \text{for } j = 1, \ldots, n-1,$$

the inequality $x \leqslant z$ follows from (c). Similarly we can get $y \leqslant z$. If $x \leqslant w$ and $y \leqslant w$, where $w = k_1 a_1 + \ldots + k_{n-1}$ is an (a)-type representation, then, by (c),

$$\sum_{i=j}^{n-1} h_i = \sum_{i=j}^{n-1} f_i \cup \sum_{i=j}^{n-1} g_i \leqslant \sum_{i=j}^{n-1} k_i \quad \text{for } j = 1, \ldots, n-1,$$

which shows that z is the least upper bound of x and y. Similarly we can prove that

(3.4) $$x \cap y = l_1 a_1 + \ldots + l_{n-1},$$

where

(3.5) $$l_i = \Big(\sum_{j=i}^{n-1} f_j \cap \sum_{j=i}^{n-1} g_j\Big) - \Big(\sum_{j=i+1}^{n-1} f_j \cap \sum_{j=i+1}^{n-1} g_j\Big)$$

for $i = 1, \ldots, n-2$ and $l_{n-1} = f_{n-1} \cap g_{n-1}$.

The distributivity of the lattice P follows, by an easy computation, from formulas (3.2), (3.5), and from the distributivity of the Boolean algebra B.

In the third step of the proof we show that condition 7° is satisfied. If $f \in B$, then $f = fa_{n-1}$ is the unique (a)-type representation of the function f. The unique (a)-type representation of the constant function a_k is $a_k = 1 \cdot a_k$. Then

$$f \cap a_k = \sum_{i=1}^{n-1} l_i a_i,$$

where, for $k = 1, \ldots, n-1$,

$$l_i = \begin{cases} 0 & \text{if } i \neq k, \\ f & \text{if } i = k. \end{cases}$$

Hence $f \cap a_k = fa_k$ for every $f \in B$ and $k = 1, \ldots, n-1$.

If $x = f_1 a_1$ and $y = g_2 a_2$ are (a)-type representations and $f_1 + g_2 \leqslant 1$, then, by (3.2),

$$x \cup y = f_1 a_1 \cup g_2 a_2 = \sum_{i=1}^{n-1} h_i a_i,$$

where

$$h_i = \begin{cases} 0 & \text{for } i > 2, \\ g_2 & \text{for } i = 2, \\ f_1 & \text{for } i = 1. \end{cases}$$

Hence it follows that $f_1 + f_2 \leqslant 1$ implies

$$f_1 a_1 \cup f_2 a_2 = f_1 a_1 + f_2 a_2 .$$

Now, let $f_1 + \ldots + f_k \leqslant 1$ and assume that, for all i,

$$x = f_1 a_1 \cup \ldots \cup f_{k-1} a_{k-1} = f_1 a_1 + \ldots + f_{k-1} a_{k-1}, \qquad f_i \in B .$$

Then, setting $y = f_k a_k$ and using formulas (3.2) and (3.3) once more, we get

$$x \cup y = h_1 a_1 + \ldots + h_{n-1},$$

where

$$h_i = \begin{cases} 0 & \text{for } i > k, \\ f_i & \text{for } i \leqslant k. \end{cases}$$

Hence

$$f_1 a_1 \cup \ldots \cup f_k a_k = f_1 a_1 + \ldots + f_k a_k \quad \text{for } k = 1, \ldots, n-1$$

by inductive arguments.

It remains to show that B is the center of all complemented elements of P, but this follows from Theorem 2.2 of [2].

The converse statement of Theorem 1 is also true. In fact, by definition, conditions I and II (b) are satisfied. Condition II (a) follows from 7°. Condition II (c), in view of 7°, is a reformulation of the well-known equivalence

$$x \leqslant y \quad \text{if and only if} \quad \bigcup_{i=k}^{n-1} C_i(x) \leqslant \bigcup_{i=k}^{n-1} C_i(y) \text{ for } k = 1, \ldots, n-1,$$

satisfied in any Post algebra (see, e.g., [1]).

4. Numerical representation theorem. Here we prove the following theorem:

THEOREM 2. *Every Post algebra can be isomorphically represented as a numerical Post algebra.*

Proof. Let $(P; e_0, \ldots, e_{n-1})$ be a Post algebra of order n with center B of complemented elements of P. Let $M = (M_B; a_0, \ldots, a_{n-1})$ be a full set of measures on P (see Section 2). Let a mapping $\bar{x}\colon M \to [0, 1]$ defined by

$$\bar{x}(m) = m(x) = m_B\big(C_1(x)\big) a_1 + \ldots + m_B\big(C_{n-1}(x)\big), \qquad m \in M, \tag{4.1}$$

be associated with any $x \in P$.

The set $\overline{P} = \{\overline{x}: x \in P\}$ is a numerical Post algebra with respect to the natural ordering of functions. To show this, let us observe firstly that the set $\overline{B} \subset \overline{P}$ of all maps $\overline{a}$, where $a \in B$, is a numerical Boolean algebra with respect to the same order. This was shown by Mączyński in [4]. Since $m(e_i) = a_i$ for every $m \in M$ and for $i = 0, 1, \ldots, n-1$, there are n constant functions in $\overline{P}$:

$$0 = \overline{e}_0,\ a_1 = \overline{e}_1, \ldots, a_{n-1} = \overline{e}_{n-1} = 1.$$

Thus it follows from (4.1) that every $\overline{x} \in \overline{P}$ has the (a)-type representation

$$\overline{x} = \overline{C_1(x)}\,a_1 + \ldots + \overline{C_{n-1}(x)}. \tag{4.2}$$

We have just shown that conditions I and II (b) from Theorem 1 are satisfied. Condition II (a) can be proved as easily as those above.

We proceed to prove II (c). Since the set M of measures on P is full, $\overline{x} \leqslant \overline{y}$ if and only if $x \leqslant y$. By the well-known property of Post algebras, $x \leqslant y$ if and only if

$$\bigcup_{j=i}^{n-1} C_j(x) \leqslant \bigcup_{j=i}^{n-1} C_j(y) \quad \text{for } i = 1, \ldots, n-1.$$

Therefore, $\overline{x} \leqslant \overline{y}$ if and only if

$$\sum_{j=i}^{n-1} m_B\big(C_j(x)\big) \leqslant \sum_{j=i}^{n-1} m_B\big(C_j(y)\big) \quad \text{for all } m_B \in M_B \text{ and } i = 1, \ldots, n-1$$

in view of the fact that M_B is a full set of measures. This equivalence, however, means that $\overline{x} \leqslant \overline{y}$ if and only if

$$\sum_{j=i}^{n-1} \overline{C_j(x)} \leqslant \sum_{j=i}^{n-1} \overline{C_j(y)} \quad \text{for } i = 1, \ldots, n-1,$$

which, by (4.2), proves II (c).

Since M is a full set of measures, the mapping h from P onto $\overline{P}$, defined by $h(x) = \overline{x}$, is one-to-one. The restriction $h|B$ is a Boolean isomorphism from B onto $\overline{B}$. Furthermore, h maps constants of the Post algebra P onto constants of the numerical Post algebra $\overline{P}$; $h(e_i) = a_i$ for $i = 0, 1, \ldots, n-1$. Therefore, h is an isomorphism (see [6]). This completes the proof of the theorem [1].

(1) If M is the set of all n-valued measures on P, then the above-described numerical representation coincides with the well-known Epstein representation [1].

5. Full sets of measures. If $P \subset [0,1]^M$ is a numerical Post algebra, then a full set of measures on P may be obtained in the following way. Let, for an $m \in M$,

$$\varphi_m(x) = f_1(m)a_1 + \ldots + f_{n-1}(m), \tag{5.1}$$

where $f_1 a_1 + \ldots + f_{n-1}$ is an (a)-type representation of $x \in P$. For every $m \in M$, φ_m is a measure on P. In fact, φ_m is, obviously, a mapping from P into $[0,1]$; also $\varphi_m(e_i) = a_i$ for $i = 0, 1, \ldots, n-1$. If $x \cap y = 0$ for some $x, y \in P$, then, by Lemma 5.3 of [6],

$$C_i(x \cup y) = C_i(x) \cup C_i(y) \quad \text{and} \quad C_i(x) \cap C_i(y) = 0 \quad \text{for } i = 1, \ldots, n-1.$$

Therefore, if $g_1 a_1 + \ldots + g_{n-1}$ is an (a)-type representation of y, we get

$$x \cup y = (f_1 + g_1)a_1 + \ldots + (f_{n-1} + g_{n-1}),$$

whence $\varphi_m(x \cup y) = \varphi_m(x) + \varphi_m(y)$ for $x \cap y = 0$. Evidently, $\{\varphi_m : m \in M\}$ is a full set of measures.

It follows from Section 4 that, in order to build up a numerical representation of a given Post algebra $(P; e_0, e_1, \ldots, e_{n-1})$, one ought to start with a full set $M = (M_B; a_0, a_1, \ldots, a_{n-1})$ of measures on P with property (2.2) for every $m \in M$.

An important question is, however, whether a given full set M_B of measures on the center B of P can induce, by definition (2.2), a full set of measures on P. We have shown in Section 2 that, by extending all two-valued measures on B in this way, we get a full set of measures on P independently of how the chain $0 = a_0 < a_1 < \ldots < a_{n-1} = 1$ has been chosen. Generally, it is not the case.

Then, we now define the numbers $a_1, \ldots, a_{n-1}$ one by one in such a way, if possible, that the set M of all extensions of form (2.2) be full. Let A_1 be the set of all numbers $a \leqslant 1$ such that, for all $x, y \in B$, $x \nleqslant y$ implies $am(x) > m(y)$ for some $m \in M_B$. Since M_B is full, $1 \in A_1$, i.e. A_1 is not empty. Evidently, $\inf A_1 \geqslant 0$. We choose an $a_1 \in A_1$, $a_1 < 1$, if possible. Assume, by induction, that the numbers $a_i \in A_i$ $(i = 1, \ldots, k-1)$ have been chosen in such a way that $a_1 < a_2 < \ldots < a_{k-1} < 1$. Let A_k be the set of all numbers $a \leqslant 1$ such that, for all $x, y \in B$, $x \nleqslant y$ implies

$$am(x) > m(y) + m(y')a_{k-1}$$

for some measure $m \in M_B$. If A_k is not empty, then $\inf A_k \geqslant a_{k-1}$. In fact, for $x = 1$ and $y = 0$, we get $a > a_{k-1}$. We choose $a_k \in A_k$, $a_{k-1} < a_k < 1$, if possible, for $k = 1, \ldots, n-2$ and $a_{n-1} = 1$.

THEOREM 3. *M_B induces a full set of measures M on the Post algebra $(P; e_0, e_1, \ldots, e_{n-1})$ if and only if, in the above-described algorithm, $a_i \in A_i$ may be chosen in such a way that $A_{i+1} \neq \emptyset$ for $i = 1, \ldots, n-2$.*

Proof. Sufficiency. Let $a_i \in A_i$ $(i = 1, \ldots, n-1)$, $a_1 < a_2 < \ldots < a_{n-1} = 1$, let

$$x = \bigcup_{i=1}^{n-1} \big(C_i(x) \cap e_i\big) \quad \text{and} \quad y = \bigcup_{i=1}^{n-1} \big(C_i(y) \cap e_i\big)$$

be disjoint representations of $x, y \in P$, and let $x \not\leqslant y$, i.e.

$$C_k(x) \not\leqslant \bigcup_{j=k}^{n-1} C_j(y) \quad \text{for some } k \leqslant n-1.$$

Since $a_k \in A_k$, there exists an $m_B \in M_B$ such that

$$m_B\big(C_k(x)\big)a_k > m_B\Big(\bigcup_{j=k}^{n-1} C_j(y)\Big) + m_B\Big(\Big(\bigcup_{j=k}^{n-1} C_j(y)\Big)'\Big)a_{k-1}.$$

If m is the extension of m_B such that (2.2) holds, then

$$m(y) = \sum_{i=1}^{n-1} m_B\big(C_i(y)\big)a_i \leqslant \sum_{i=1}^{k-1} m_B\big(C_i(y)\big)a_{k-1} + \sum_{i=k}^{n-1} m_B\big(C_i(y)\big)$$

$$\leqslant m_B\Big(\Big(\bigcup_{i=k}^{n-1} C_i(y)\Big)'\Big)a_{k-1} + m_B\Big(\bigcup_{i=k}^{n-1} C_i(y)\Big) < m_B\big(C_k(x)\big)a_k \leqslant m(x).$$

Hence $M = (M_B; a_0, a_1, \ldots, a_{n-1})$ is a full set of measures.

Necessity. Suppose that $A_k = \emptyset$ for certain $k \leqslant n-1$ and for every choice of $a_i \in A_i$, $i < k$. Then for every number a_k there exist $x, y \in B$ such that $x \not\leqslant y$ and

$$m_B(x)a_k \leqslant m_B(y) + m_B(y')a_{k-1} \quad \text{for all } m_B \in M_B.$$

Therefore, by (2.2),

$$m(x \cap e_k) \leqslant m(y' \cap e_{k-1} \cup y) \quad \text{for all } m \in M.$$

But $x \cap e_k \not\leqslant y' \cap e_{k-1} \cup y$, which shows that the set M of measures is not full.

Examples. Consider a 4-element Boolean algebra $B = \{0, a, a', 1\}$ Let m_1 and m_2 be two measures on B defined by the following table:

	0	a	a'	1
m_1	0	2/5	3/5	1
m_2	0	3/5	2/5	1

$M_B = \{m_1, m_2\}$ is a full set of measures on B. A simple computation shows that $A_1 \neq \emptyset$ and $A_2 = \emptyset$. Hence M_B does not induce a full set of measures on any Post algebra with center B. If, however, m_1 and m_2 are defined by the table

	0	a	a'	1
m_1	0	1/4	3/4	1
m_2	0	3/4	1/4	1

then $A_1 \neq \emptyset$, $A_2 \neq \emptyset$ for $1/3 < a_1 < 2/3$, and $A_3 = \emptyset$. Therefore, this time M_B induces a full set of measures on a Post algebra of order 3, but it does not induce a full set of measures on any Post algebra of order greater than 3.

LEMMA 1. *If M_B is a full set of measures on a Boolean algebra B, then for every $x \in B$, $x \neq 0$, there exists a measure $m_B \in M_B$ such that $m_B(x) > 1/2$.*

Proof. If, on the contrary, $m_B(x) \leqslant 1/2$ for all $m_B \in M_B$, then $m_B(x) \leqslant m_B(x')$ for all $m_B \in M_B$, which implies $x \leqslant x'$, a contradiction.

THEOREM 4. *Assume that the Post algebra $(P; e_0, e_1, \ldots, e_{n-1})$ is finite, and that M_B is a full set of measures on the center B of P. Then $M = (M_B; a_0, a_1, \ldots, a_{n-1})$ is a full set of measures on P if and only if*

$$a_k > a_{k-1} + \frac{1-\varepsilon}{\varepsilon} \quad \textit{for } k = 1, \ldots, n-1,$$

where

$$\varepsilon = \inf_{0 \neq x \in B} \sup_{m_B \in M_B} m_B(x).$$

Proof. Sufficiency. Let

$$x = \bigcup_{j=1}^{n-1} \big(C_j(x) \cap e_j\big) \quad \text{and} \quad y = \bigcup_{j=1}^{n-1} \big(C_j(y) \cap e_j\big)$$

be two disjoint representations of $x, y \in P$. If $x \not\leqslant y$, then, for certain k,

$$C_k(x) \not\leqslant \bigcup_{j=k}^{n-1} C_j(y).$$

Therefore, there exists a non-zero element $u \in B$ such that

$$u \leqslant C_k(x) \quad \text{and} \quad u \leqslant \Big(\bigcup_{j=k}^{n-1} C_j(y)\Big)'.$$

Let $m \in M$ be a measure such that $m|B = m_B^u$, where $m_B^u(u) \geqslant \varepsilon > 1/2$ (see Lemma 1). Then

$$m(y) \leqslant m_B^u\Big(\Big(\bigcup_{j=k}^{n-1} C_j(y)\Big)'\Big) a_{k-1} + m_B^u\Big(\bigcup_{j=k}^{n-1} C_j(y)\Big) \leqslant m_B^u(u)\, a_{k-1} + m_B^u(u').$$

The last inequality is a consequence of the simple arithmetical fact that $0 \leqslant a, b, c \leqslant 1$ and $a \leqslant b$ imply $bc + 1 - b \leqslant ac + 1 - a$.

Consequently,

$$m(y) \leqslant m_B^u(u)\, a_{k-1} + \frac{1-\varepsilon}{\varepsilon} m_B^u(u) < m_B^u(u)\, a_k \leqslant m_B^u\big(C_k(x)\big) a_k \leqslant m(x).$$

Thus the set M is full.

Necessity. Suppose that, for some k,

$$a_k \leqslant a_{k-1} + \frac{1-\varepsilon}{\varepsilon}, \quad \text{i.e.} \quad \varepsilon a_k \leqslant \varepsilon a_{k-1} + 1 - \varepsilon.$$

Since the set M_B is finite, we can choose an atom $u_0 \in B$ and a measure $m_B^{(0)} \in M_B$ such that $m_B^{(0)}(u_0) = \varepsilon$. Since the inequality $m_B(u_0) \leqslant \varepsilon$ holds for all $m_B \in M_B$, we have

$$m_B(u_0)a_k \leqslant \varepsilon a_k \leqslant \varepsilon a_{k-1} + 1 - \varepsilon \leqslant m_B(u_0)a_{k-1} + m_B(u_0') \quad \text{for all } m_B \in M_B.$$

This proves that

$$m(u_0 \cap e_k) \leqslant m(u_0 \cap e_{k-1} \cup u_0') \quad \text{for all } m \in M.$$

But $u_0 \cap e_k \not\leqslant u_0 \cap e_{k-1} \cup u_0'$, contradicting the fullness of M. In the above-considered examples, for the former we have $\varepsilon = 3/5$, and for the latter we have $\varepsilon = 3/4$.

We finish the paper with the following property of the family of all full sets of measures on a given Boolean algebra:

THEOREM 5. *For every full set M_B of measures on an atomic Boolean algebra B there exists a subset $M_B' \subset M_B$ which is minimal in the family (ordered by inclusion) of all full sets of measures on B. The cardinal number of M_B' equals the one of the set of atoms of the Boolean algebra B.*

Proof. Let A be the set of all atoms of B. For each $x \in A$ let us choose a measure $m_B^x \in M_B$ such that $m_B^x(x) > 1/2$. We assert that $M_B' = \{m_B^x \in M_B \colon x \in A\}$ is a full set of measures. In fact, assume that $y_1 \not\leqslant y_2$ for some $y_1, y_2 \in B$. Then there exists an $x \in A$ such that $x \leqslant y_1$ and $y_2 \leqslant x'$. Hence

$$m_B^x(y_1) \geqslant m_B^x(x) > \frac{1}{2} > m_B^x(x') \geqslant m_B^x(y_2),$$

which implies the fullness of M_B'.

Now consider $M_B'' = M_B' \setminus \{m_B^{x_0}\}$ for a fixed atom $x_0 \in A$. Then, for all measures $m_B^x \in M_B''$ we have $m_B^x(x_0) < 1/2$ (since $x \cap x_0 = 0$ and $m_B^x(x) > 1/2$). Therefore, for all $m_B^x \in M_B''$,

$$m_B^x(x_0) < \frac{1}{2} < m_B^x(x_0'),$$

but $x_0 \not\leqslant x_0'$, i.e. M_B'' is not full.

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