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ON TOPOLOGICAL TYPES OF THE SIMPLEST INDECOMPOSABLE CONTINUA

BY

W. DĘBSKI (KATOWICE)

Indecomposable (*metric*) *continua of the simplest type* are those, by the definition, which can be obtained as inverse limits of sequences $I \xleftarrow{p_1^2} I \xleftarrow{p_2^3} I \xleftarrow{p_3^4} \ldots$, where I are the unit intervals, $I = [0, 1] = \{t\colon 0 \leqslant t \leqslant 1\}$, and p_n^{n+1} are open continuous maps, i.e. continuous maps which are strictly monotone on intervals and have local extrema equal to 0 and 1; the number of intervals of monotonicity will be called the *degree* of such a map.

Rogers, Jr. [3] proved that each continuum of the simplest type can be mapped onto another one.

The aim of this paper* is to show that, in contrast to Rogers' result, there exist $2^{\aleph_0}$ continua of the simplest type such that no one is the open continuous image of the other. In particular: there exist $2^{\aleph_0}$ topologically different continua of the simplest type. It is shown how the topological type depends on the sequence of degrees of maps $p_1^2, p_2^3, \ldots$ in the inverse sequence. This dependence is similar to that of Cook's topological classification of solenoids [1].

1. Preliminaries. For each positive integer n we consider the standard open (continuous) map $w_n\colon I \to I$ on the unit interval $I = \{t\colon 0 \leqslant t \leqslant 1\}$, namely a map defined by

$$w_n\left(\frac{i}{n}\right) = \begin{cases} 0 & \text{if } i \text{ is even,} \\ 1 & \text{if } i \text{ is odd,} \end{cases}$$

which is linear on closed intervals $[(i-1)/n, i/n]$.

We have $w_n \circ w_m = w_{nm}$.

LEMMA 1. *If $p\colon I \to I$ is an arbitrary open map on the unit interval, then there exist a number n and a sequence $0 = a_0 < a_1 < \ldots < a_n = 1$ such that p restricted to the closed interval $[a_{i-1}, a_i]$ is a homeomorphism onto I.*

* This paper is a part of the author's doctoral dissertation [2].

The number n, uniquely determined by p, will be called the *degree* of p; we write $n = \deg p$.

Let us observe that if p and q are open maps from I onto I, then $\deg p \circ q = (\deg p)(\deg q)$.

Clearly, $\deg w_n = n$.

LEMMA 2. *If open maps p and q on I are such that $|p(x)-q(x)| \leqslant \frac{1}{2}$ for each x, then $\deg p = \deg q$.*

Proof. Let us consider maximal intervals, with respect to inclusion, on which p and q are both increasing or both decreasing. It is easy to see that the number of those intervals is equal to $\deg p$ and to $\deg q$.

Using Lemma 1 we get the following

LEMMA 3. *If open maps p and q on I are such that $\deg p = \deg q$ and $p(x) = q(y)$, where x and y are taken from the set of ends of I, then there exists a unique homeomorphism $h\colon I \to I$ such that $p \circ h = q$ and $x = h(y)$.*

LEMMA 4. *Let $I_1 \xleftarrow{p_1^2} I_2 \xleftarrow{p_2^3} I_3 \xleftarrow{p_3^4} \ldots$ and $I_1 \xleftarrow{q_1^2} I_2 \xleftarrow{q_2^3} I_3 \xleftarrow{q_3^4} \ldots$, where $I_n = I$, be inverse sequences consisting of open maps on I such that $\deg p_i^{i+1} = \deg q_i^{i+1}$ for each i. Then there exists a homeomorphism between the limits of those sequences.*

Proof. Let $x_1, x_2, \ldots$ be a sequence consisting of ends of I such that $x_i = p_i^{i+1}(x_{i+1})$ for each i; let $y_1, y_2, \ldots$ be a sequence for the maps q_i^{i+1} defined analogously. We define inductively homeomorphisms $h_i\colon I_i \to I_i$ as follows.

Put $h_1(t) = t$ if $x_1 = y_1$ and $h_1(t) = 1-t$ if $x_1 \neq y_1$.

If $h_i\colon I_i \to I_i$ is given, then h_{i+1} is uniquely determined by the conditions $p_i^{i+1} \circ h_{i+1} = h_i \circ q_i^{i+1}$ and $x_{i+1} = h_{i+1}(y_{i+1})$, according to Lemma 3. Now, the homeomorphisms h_i induce a homeomorphism between limits.

From Lemma 4 it follows that a space obtained as an inverse limit of the sequence of closed intervals and open maps is homeomorphic to an inverse limit of an appropriate sequence consisting of standard open maps.

Lemmas 1 and 4 can be found also in [3].

2. Approximation lemma. We begin with a lemma concerning maps on the interval.

LEMMA 5. *Let ε, $0 < \varepsilon < \frac{1}{2}$, be given. Let $m_1, \ldots, m_n$, $M_1, \ldots, M_n$ be numbers from $I = [0, 1]$ such that*

A1. *$m_i < M_i$;*

A2. *$m_i \leqslant M_{i+1}$ and $M_i \geqslant m_{i+1}$ (this means that the intersection of segments $[m_i, M_i]$ and $[m_{i+1}, M_{i+1}]$ is non-empty);*

A3. *$M_i \leqslant \varepsilon$ for some i;*

A4. *$m_i \geqslant 1-\varepsilon$ for some i;*

A5. *$M_1 \leqslant \varepsilon$ or $m_1 \geqslant 1-\varepsilon$;*

A6. *$M_n \leqslant \varepsilon$ or $m_n \geqslant 1-\varepsilon$;*

A7. *if* $M_i > \varepsilon$ *and* $m_i < 1-\varepsilon$, *then*

$$M_{i-1} < M_i < M_{i+1} \quad \text{and} \quad m_{i-1} < m_i < m_{i+1}$$

or

$$M_{i-1} > M_i > M_{i+1} \quad \text{and} \quad m_{i-1} > m_i > m_{i+1}.$$

Then there exists an open map $g\colon I \to I$ *such that if* $x \in [(i-1)/n, i/n]$, *then*

$$g(x) \leqslant \varepsilon \text{ if } M_i \leqslant \varepsilon, \quad g(x) \geqslant 1-\varepsilon \text{ if } m_i \geqslant 1-\varepsilon,$$

$$m_i \leqslant g(x) \leqslant M_i \quad \text{if } M_i > \varepsilon \text{ and } m_i < 1-\varepsilon.$$

Proof. Let c_i be the center of the interval $[m_i, M_i] \cap [m_{i+1}, M_{i+1}]$ mentioned in A2.

If $M_i > \varepsilon$ and $m_i < 1-\varepsilon$, then we define g on $[(i-1)/n, i/n]$ to be a linear function such that $g\big((i-1)/n\big) = c_{i-1}$ and $g(i/n) = c_i$.

Now we define g on those segments $[(i-1)/n, i/n]$ where m_i and M_i are both less than ε or both greater than $1-\varepsilon$.

If $M_1 \leqslant \varepsilon$, then take k such that $M_i \leqslant \varepsilon$ for $i \leqslant k$ and $M_{k+1} > \varepsilon$. We define g on $[0, k/n]$ to be a linear map such that $g(0) = 0$ and $g(k/n) = c_k$.

If $m_1 \geqslant 1-\varepsilon$, then take k such that $m_i \geqslant 1-\varepsilon$ for $i \leqslant k$ and $m_{k+1} < 1-\varepsilon$. We define g on $[0, k/n]$ to be a linear map such that $g(0) = 1$ and $g(k/n) = c_k$.

If $M_n \leqslant \varepsilon$, then take k such that $M_i \leqslant \varepsilon$ for $i \geqslant k$ and $M_{k-1} > \varepsilon$. We define g on $[(k-1)/n, 1]$ to be a linear map such that $g\big((k-1)/n\big) = c_{k-1}$ and $g(1) = 0$.

If $m_n \geqslant 1-\varepsilon$, then take k such that $m_i \geqslant 1-\varepsilon$ for $i \geqslant k$ and $m_{k-1} < 1-\varepsilon$. We define g on $[(k-1)/n, 1]$ to be a linear map such that $g\big((k-1)/n\big) = c_{k-1}$ and $g(1) = 1$.

If k and l are such that $k \leqslant l$, $M_i \leqslant \varepsilon$ if $k \leqslant i \leqslant l$, $M_{k-1} > \varepsilon$, and $M_{l+1} > \varepsilon$, then we define g on the segment $[(k-1)/n, l/n]$ to be a linear map on each half of that segment such that

$$g\left(\frac{k-1}{n}\right) = c_{k-1}, \quad g\left(\frac{l+k-1}{2n}\right) = 0, \quad \text{and} \quad g\left(\frac{l}{n}\right) = c_l.$$

If k and l are such that $k \leqslant l$, $m_i \geqslant 1-\varepsilon$ for $k \leqslant i \leqslant l$, $m_{k-1} < 1-\varepsilon$, and $m_{l+1} < \varepsilon$, then we define g on the segment $[(k-1)/n, l/n]$ to be a linear map on each half of that segment such that

$$g\left(\frac{k-1}{n}\right) = c_{k-1}, \quad g\left(\frac{l+k-1}{2n}\right) = 1, \quad \text{and} \quad g\left(\frac{l}{n}\right) = c_l.$$

Thus, we have defined a map $g\colon I \to I$ which, by A3-A7, is increasing or decreasing on the intervals, and at the ends of those intervals admits the different values 0 and 1. Therefore, g is open. It follows from the construction that g satisfies the remaining conditions.

LEMMA 6. *Let* ε, $0 < \varepsilon < \frac{1}{2}$, *be given. Let* $m_1, \ldots, m_n$, $M_1, \ldots, M_n$ *be numbers from* $I = [0, 1]$ *such that*

B1. $0 < M_i - m_i \leqslant \varepsilon$;

B2. $m_i \leqslant M_{i+1}$ *and* $M_i \geqslant m_{i+1}$;

B3. *if* $M_i \geqslant M_j$ *for indices* j *adjacent to* i, *then* $M_i = 1$;

B4. *if* $m_i \leqslant m_j$ *for indices* j *adjacent to* i, *then* $m_i = 0$.

Then conditions A1-A7 (*from Lemma* 5) *are satisfied.*

Proof. Conditions A1 and A2 are consequences of B1 and B2.

If $m_i \leqslant m_j$ for each j (i.e., if m_i is the minimum of $m_1, \ldots, m_n$), then, by B4, $m_i = 0$ and, by B1, $M_i \leqslant \varepsilon$. This means that condition A3 is satisfied.

Analogously, condition A4 is satisfied.

Now we show that

B5. *there exists no* i *such that*

(*) $M_i > \varepsilon$, $m_i < 1-\varepsilon$, $M_i \leqslant M_{i+1}$, *and* $m_i \geqslant m_{i+1}$

and

B6. *there exists no* i *such that*

(**) $M_i > \varepsilon$, $m_i < 1-\varepsilon$, $M_{i-1} \geqslant M_i$, *and* $m_{i-1} \leqslant m_i$.

By symmetry, it suffices to show B5 only.

Suppose (*) holds for some i. We have $m_{i+1} < 1-\varepsilon$. Since, by B1, $M_{i+1} - m_{i+1} \leqslant \varepsilon$, we get $M_{i+1} < 1$. Analogously, $m_{i+1} > 0$. If $i+1 < n$, then $M_{i+1} \leqslant M_{i+2}$, since otherwise M_{i+1} should be a local maximum, and therefore, by B3, we would have $M_{i+1} = 1$; analogously, $m_{i+1} \geqslant m_{i+2}$ and, as before, $M_{i+2} < 1$ and $m_{i+2} > 0$. By induction, we get

$$M_i \leqslant M_{i+1} \leqslant \ldots \leqslant M_n < 1 \quad \text{and} \quad m_i \geqslant m_{i+1} \geqslant \ldots \geqslant m_n > 0,$$

which contradicts B3 and B4.

Suppose $M_1 > \varepsilon$ and $m_1 < 1-\varepsilon$. Since, by B3, $M_1 \leqslant M_2$ and, by B4, $m_1 \geqslant m_2$, we get a contradiction by B5. This means that condition A5 is satisfied.

Analogously, condition A6 is satisfied.

Condition A7 is implied by the fact that if $M_i > \varepsilon$ and $m_i < 1-\varepsilon$, then the following four possibilities are excluded by B3-B6:

$$M_{i-1} \leqslant M_i \geqslant M_{i+1}, \quad m_{i-1} \geqslant m_i \leqslant m_{i+1},$$

$$M_i \leqslant M_{i+1} \text{ and } m_i \geqslant m_{i+1}, \quad M_{i-1} \geqslant M_i \text{ and } m_{i-1} \leqslant m_i.$$

APPROXIMATION LEMMA. *Let* K *be the inverse limit of the sequence* $I_1 \xleftarrow{p_1^2} I_2 \xleftarrow{p_2^3} I_3 \xleftarrow{p_3^4} \ldots$, *where* $I_j = I$, *consisting of continuous maps of* I *onto* I. *Let* $p_j\colon K \to I_j$ *be the projections. Let* $f\colon K \to I$ *be an open and continuous map onto the closed interval* I. *Then for every* $\varepsilon > 0$ *there exists* l_0 *such that if* $l \geqslant l_0$, *then there exists an open and continuous map* $g\colon I_l \to I$ *such that* $|f(x) - g(p_l(x))| < \varepsilon$ *for each* x.

Proof. Assume that $\varepsilon > \frac{1}{2}$. There exists $\mu > 0$ such that if $\varrho(x_1, x_2) < \mu$, then $|f(x_1)-f(x_2)| < \varepsilon$ (here ϱ is a metric inducing the topology on K). Let l_0 be an integer such that if $l \geqslant l_0$, then the diameters of $p_l^{-1}(x)$ are less than μ. Hence, for each $l \geqslant l_0$ there exists $\delta = \delta(l) > 0$ such that if $|p_l(x_1)-p_l(x_2)| < \delta$, then $\varrho(x_1, x_2) < \mu$. Thus, if $|p_l(x_1)-p_l(x_2)| < \delta$, then $|f(x_1)-f(x_2)| < \varepsilon$.

Let l, $l \geqslant l_0$, and $\delta = \delta(l)$ be chosen for a given $\varepsilon > 0$ so that the above conditions are satisfied.

Let n be such that $1/n < \delta$. Let $0 < 1/n < \ldots < (n-1)/n < 1$ be a partition of the unit interval I_l.

For $0 < i \leqslant n$ we put

$$M_i = \sup f\left(p_l^{-1}\left(\left[\frac{i-1}{n}, \frac{i}{n}\right]\right)\right), \qquad m_i = \inf f\left(p_l^{-1}\left(\left[\frac{i-1}{n}, \frac{i}{n}\right]\right)\right).$$

Now we show that the assumptions of Lemma 6 are satisfied. By Lemma 6, the assumptions of Lemma 5 will be also satisfied.

Obviously, $0 \leqslant M_i - m_i \leqslant \varepsilon$. Since the set $A = f\left(p_l^{-1}\left([(i-1)/n, i/n]\right)\right)$ contains the open and non-empty set $f\left(p_l^{-1}\left(((i-1)/n, i/n)\right)\right)$, the map p_l is onto and continuous and f is open, we infer that A consists of more than one point and $0 < M_i - m_i$. This means that condition B1 is satisfied.

Since $p_l(x) = i/n$ for some x, we have $m_i \leqslant f(x) \leqslant M_{i+1}$ and $M_i \geqslant f(x) \geqslant m_{i+1}$. Thus, condition B2 is satisfied.

Let $M_i \geqslant M_j$ for indices j adjacent to i. Then the supremum M_i of the compact non-empty set A belongs to A. Since A is contained in the set

$$B = f\left(p_l^{-1}\left(\left(\frac{i-2}{n}, \frac{i+1}{n}\right) \cap I\right)\right)$$

and $M_i \geqslant M_j$ for indices j adjacent to i, it follows that $M_i = \sup B$ belongs to B, which is possible only in the case where $M_i = 1$ as B is open in I. This means that condition B3 is satisfied.

Analogously, condition B4 is satisfied.

The function g, whose existence is assured by Lemma 5, is the desired one.

Let K be the inverse limit of the sequence $I_1 \xleftarrow{p_1^2} I_2 \xleftarrow{p_2^3} I_3 \xleftarrow{p_3^4} \ldots$ Let p_j: $K \to I_j$ denote the projection. Let f: $K \to I$ and f_i: $I_i \to I$. If $f_i \circ p_i$ is uniformly convergent to f, then we say that f_i is an *approximating sequence* for f.

The following corollary is a consequence of the Approximation Lemma.

COROLLARY. *Let K be the inverse limit of the sequence $I_1 \xleftarrow{p_1^2} I_2 \xleftarrow{p_2^3} I_3 \xleftarrow{p_3^4} \ldots$ consisting of continuous maps of $I_j = I$ onto I. Let f: $K \to I$ be an open and continuous map onto I. Then there exist open and continuous maps f_i: $I_i \to I$ such that f_i is an approximating sequence for f.*

3. Open maps from inverse limits. Let p be an inverse system $I_1 \xleftarrow{p_1^2} I_2 \xleftarrow{p_2^3} I_3 \xleftarrow{p_3^4} \ldots$ consisting of standard open maps of the interval $I = [0, 1]$

into itself. Let P be the limit of p. By p_i: $P \to I_i$ we denote the projections.

LEMMA 7. *Let* h: $I_i \to I$ *and* g: $I_j \to I$ *be open maps such that*

$$|h(p_i(x))-g(p_j(x))| \leqslant \tfrac{1}{2} \quad \textit{for all } x \in P.$$

Then

$$\frac{\deg h}{\deg p_1^i} = \frac{\deg g}{\deg p_1^j}.$$

Proof. Assume that $j \geqslant i$. We have $|h(p_i^j(x))-g(x)| \leqslant \frac{1}{2}$ for $x \in I_j$. By Lemma 2, $\deg h \circ p_i^j = \deg g$. Since $\deg h \circ p_i^j = (\deg h)(\deg p_i^j)$ and $\deg p_1^j = (\deg p_1^i)(\deg p_i^j)$, we have the desired equality.

Let f: $P \to I$ be open. Let f_i: $I_i \to I$ be an approximating sequence for f. By Lemma 7, the numbers

$$(***) \qquad \frac{\deg f_i}{\deg p_1^i}$$

are equal for almost all i (as $|f(x)-f_i(p_i(x))| \leqslant \frac{1}{4}$ for sufficiently large i). By the same lemma, the stabilized value of (***) is independent of the choice of the approximating sequence.

Define $\deg f$, the degree of f with respect to the given expansion p, to be the common stabilized value of (***) for approximating sequences of f.

From the definition it follows that $(\deg f)(\deg p_1^i)$ is an integer if i is sufficiently large.

LEMMA 8. *Let* f: $P \to I$ *and* g: $I \to I$ *be open maps. Then*

$$\deg g \circ f = (\deg g)(\deg f).$$

LEMMA 9. *Let* f: $P \to I$ *and* g: $P \to I$ *be open maps and let* $|f(x)-g(x)| < \frac{1}{2}$ *if* $x \in P$. *Then* $\deg f = \deg g$.

Let q be an inverse system $I_1 \xleftarrow{q_1^2} I_2 \xleftarrow{q_2^3} I_3 \xleftarrow{q_3^4} \ldots$ consisting of standard open maps of the interval I into itself. Let Q be the limit of q. By q_i: $Q \to I_i$ we denote the projections.

Let f: $P \to Q$ be an open map. The map $q_1 \circ f$ is also open.

We define the degree of f as the degree of $q_1 \circ f$.

LEMMA 10. *Let* f: $P \to Q$ *be an open map. Then*

$$\deg f \frac{\deg p_1^i}{\deg q_1^n}$$

is an integer if i is sufficiently large (n *fixed*).

Proof. Since

$$\deg f = \deg q_1 \circ f = \deg q_1^n \circ q_n \circ f = (\deg q_1^n)(\deg q_n \circ f)$$

(which follows from Lemma 8) and $(\deg p_1^i)(\deg q_n \circ f)$ is an integer for sufficiently large i, we have the thesis.

THEOREM. *There exist $2^{\aleph_0}$ continua of the simplest type not homeomorphic to each other.*

Proof. Observe that if the degrees of open standard maps p_i^{i+1} and q_i^{i+1} are prime numbers, $\deg p_i^{i+1} = \deg q_j^{j+1}$ only for a finite number of pairs of indices i and j, $\deg p_i^{i+1} \neq \deg p_j^{j+1}$ if $i \neq j$ and $\deg q_i^{i+1} \neq \deg q_j^{j+1}$ if $i \neq j$, then there exists no open map between the limits P and Q.

Since there exist $2^{\aleph_0}$ infinite subsets of prime numbers such that the intersection of each two of them is a finite set, there exist $2^{\aleph_0}$ continua of the simplest type which cannot be mapped by an open map one onto another and which, in consequence, are not homeomorphic.

4. Classification of continua of the simplest type. We start with some lemmas concerning the degree of open maps.

LEMMA 11. $\deg \mathrm{id}_P = 1$.

Let u be an inverse system $I_1 \xleftarrow{u_1^2} I_2 \xleftarrow{u_2^3} I_3 \xleftarrow{u_3^4} \dots$ consisting of standard open maps of the interval I into itself. Let U be the limit of u. By $u_i\colon U \to I_i$ we denote the projections.

LEMMA 12. *Let $f\colon P \to Q$ and $g\colon Q \to U$ be open maps. Then*

$$\deg g \circ f = (\deg g)(\deg f).$$

Proof. Let $g_i\colon I_i \to I$ be an approximating sequence for $u_1 \circ g$. By Lemmas 8 and 9, we have

$$\deg g \circ f = \deg u_1 \circ g \circ f = \deg g_i \circ q_i \circ f = (\deg g_i)(\deg q_i \circ f)$$
$$= \frac{\deg g_i}{\deg q_1^i} \deg q_1^i \circ q_i \circ f = \frac{\deg g_i}{\deg q_1^i} \deg q_1 \circ f = (\deg g)(\deg f).$$

LEMMA 13. *If $f\colon P \to Q$ is a homeomorphism, then*

$$(\deg f)(\deg f^{-1}) = 1.$$

LEMMA 14. *If $f\colon P \to Q$ is a homeomorphism, then*

$$\deg f \frac{\deg p_1^i}{\deg q_1^n}$$

is an integer if i is sufficiently large (n fixed) and

$$\frac{1}{\deg f} \frac{\deg q_1^n}{\deg p_1^i}$$

is an integer if n is sufficiently large (i fixed).

By Lemma 14, if P and Q are homeomorphic, then there exists a real number $r > 0$ such that

(1) $r\dfrac{\deg p_1^i}{\deg q_1^n}$ is an integer if i is sufficiently large (n fixed),

(2) $\dfrac{1}{r}\dfrac{\deg q_1^n}{\deg p_1^i}$ is an integer if n is sufficiently large (i fixed).

Remark. The condition "there exists a real number $r > 0$ such that (1) holds" is equivalent to the condition "$\deg q_1^2, \deg q_2^3, \ldots$ is a factorant of $\deg p_1^2, \deg p_2^3, \ldots$" and the condition "there exists a real number $r > 0$ such that (2) holds" is equivalent to the condition "$\deg p_1^2, \deg p_2^3, \ldots$ is a factorant of $\deg q_1^2, \deg q_2^3, \ldots$" (see [1], p. 236).

Now, we shall show that the converse is also true.

If $i_1 < i_2 < i_3 < \ldots$, then the inverse sequence

$$I \xleftarrow{p_{i_1}^{i_2}} I \xleftarrow{p_{i_2}^{i_3}} I \xleftarrow{p_{i_3}^{i_4}} \ldots$$

is said to be a *consolidation* of an inverse sequence p.

LEMMA 15. *Let p and q be inverse sequences consisting of standard open maps of the interval I such that*

(A) *there exists $r > 0$ such that* (1) *and* (2) *hold.*

Then there exists an inverse sequence v consisting of standard open maps of the interval I such that p and v, as well as q and v, have common consolidations.

Proof. Let $m_1 < m_2 < \ldots$ and $n_1 < n_2 < \ldots$ be sequences of natural numbers such that

$$a_k = r\frac{\deg p_1^{m_k}}{\deg q_1^{n_k}} \quad \text{and} \quad b_k = \frac{1}{r}\frac{\deg q_1^{n_{k+1}}}{\deg p_1^{m_k}}$$

are integers, where r is the positive number whose existence is assumed in (A); the sequences $m_1 < m_2 < \ldots$ and $n_1 < n_2 < \ldots$ can be defined inductively according to (A).

We have

$$a_k b_k = \deg q_{n_k}^{n_{k+1}} \tag{3}$$

and

$$b_k a_{k+1} = \deg p_{m_k}^{m_{k+1}}. \tag{4}$$

Consider the inverse sequence v as in the diagram

$$I_1 \xleftarrow{w_{a_1}} I_2 \xleftarrow{w_{b_1}} I_3 \xleftarrow{w_{a_2}} I_4 \xleftarrow{w_{b_2}} \ldots$$

The consolidation

$$I_1 \xleftarrow{w_{a_1} \circ w_{b_1}} I_3 \xleftarrow{w_{a_2} \circ w_{b_2}} I_5 \xleftarrow{w_{a_3} \circ w_{b_3}} \ldots$$

of v is, by (3), also a consolidation of q.

The consolidation

$$I_2 \xleftarrow{w_{b_1} \circ w_{a_2}} I_4 \xleftarrow{w_{b_2} \circ w_{a_3}} I_6 \xleftarrow{w_{b_3} \circ w_{a_4}} \ldots$$

of v is, by (4), also a consolidation of p.

Obviously, the inverse limit of a sequence and the inverse limit of a consolidation of this sequence are homeomorphic.

Since each standard open map can be decomposed into standard open maps having prime numbers as the degrees, we can restrict our considerations, without loss of generality, to the case where inverse sequences consist of standard open maps having prime numbers as the degrees.

THEOREM. *Let p and q be inverse sequences consisting of standard open maps of the interval I having prime numbers as the degrees. Then the limits P and Q are homeomorphic if and only if the following conditions hold:*

(a) *for each prime number k the set $\{i\colon \deg p_i^{i+1} = k\}$ is finite if and only if the set $\{i\colon \deg q_i^{i+1} = k\}$ is finite;*

(b) *the number of elements in $\{i\colon \deg p_i^{i+1} = k\}$ and $\{i\colon \deg q_i^{i+1} = k\}$ is the same for all but a finite number of prime numbers k.*

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INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
KATOWICE

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