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UNIQUENESS OF POSITIVE SOLUTIONS OF PARABOLIC EQUATIONS WITH UNBOUNDED COEFFICIENTS

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Introduction. In this note we shall prove the uniqueness of a non-negative solution of the Cauchy problem for the general second order linear parabolic equation with smooth but unbounded coefficients. The uniqueness of non-negative solutions of parabolic equations has been considered in several papers beginning with the well-known work of Widder [9] on the equation of heat conduction. Various extensions of Widder's result to more general parabolic equations can be found in papers [1], [6], [7], and [8]. In all of these papers, however, the coefficients of the equation are assumed to be bounded. Here we shall deal with equations whose coefficients grow at infinity in various ways.

To prove our main result we use several auxiliary theorems. The first of these is a maximum principle for parabolic equations with unbounded coefficients recently proved by Bodanko in paper [5]. Moreover, we use certain properties of the fundamental solution of such equations which were derived in paper [3] and a uniqueness theorem which was proved in paper [2]. It turns out that the required result from paper [2] is not sharp enough in one of the extreme cases for application to the problem which we treat here. In the appendix to this note we prove the appropriate sharpened version of this theorem. The same method also yields an improved version of Bodanko's theorem and of the results of paper [3]. Although these improved results are not actually needed for our work, we include their proofs in the appendix. The key new result which is needed for the proof of our uniqueness theorem is an estimate from below for the fundamental solution which generalizes an estimate derived in [4].

We consider the differential operator

$$Lu \equiv a_{ij}(x,t)u_{x_ix_j}+b_i(x,t)u_{x_i}+c(x,t)u-u_t$$

in the strip $S = E^n\times(0,T]$ for some fixed $T>0$, where E^n is the Euclidean

n-space. Here and throughout this note we use the usual summation convention. We make the following assumptions.

(i) The coefficients a_{ij}, b_i, c and their derivatives a_{ij,x_i}, a_{ij,x_ix_j}, b_{i,x_i} are locally Hölder continuous in $\bar{S}$. For some λ, $0 \leqslant \lambda \leqslant 2$, there exist positive constants K_1, K_2 and K_3 such that

$$|a_{ij}| \leqslant K_1(|x|^2+1)^{(2-\lambda)/2}, \qquad |a_{ij,x_i}|,\ |b_i| \leqslant K_2(|x|^2+1)^{1/2},$$
$$|a_{ij,x_ix_j}|,\ |b_{i,x_i}|,\ |c| \leqslant K_3(|x|^2+1)^{\lambda/2}$$

for all $(x, t) \in \bar{S}$.

(ii) There exists a constant $\gamma > 0$ such that

$$a_{ij}(x, t)\beta_i\beta_j \geqslant \gamma(|x|^2+1)^{(2-\lambda)/2}|\beta|^2$$

for all $(x, t) \in \bar{S}$ and $\beta = (\beta_1, \ldots, \beta_n) \in E^n$.

Note that if we write the operator adjoint to L in the form

$$\tilde{L}u \equiv a_{ij}(\xi, \tau)u_{\xi_i\xi_j} + \tilde{b}_i(\xi, \tau)u_{\xi_i} + \tilde{c}(\xi, \tau)u + u_\tau,$$

then it follows from (i) that

$$(1) \qquad |\tilde{b}_i| \leqslant \tilde{K}_2(|\xi|^2+1)^{1/2}, \qquad |\tilde{c}| \leqslant \tilde{K}_3(|\xi|^2+1)^{\lambda/2}$$

for some positive constants $\tilde{K}_2$ and $\tilde{K}_3$.

A function $u = u(x, t)$ is said to be a *regular solution* of $Lu = 0$ (or $\leqslant 0$) if u is continuous in $\bar{S} = E^n \times [0, T]$, and if the derivatives of u which appear in L are continuous and satisfy $Lu = 0$ (or $\leqslant 0$) at every point of S.

We shall prove the following

THEOREM. *Suppose that the coefficients of L satisfy* (i) *and* (ii). *If $u_1(x, t)$ and $u_2(x, t)$ are two non-negative regular solutions of $Lu = 0$ in S and if $u_1(x, 0) = u_2(x, 0)$ for all $x \in E^n$, then $u_1(x, t) = u_2(x, t)$ in S.*

Auxiliary results. We shall use certain results from [2], [3] and [5]. For the convenience of the reader we list them here.

THEOREM B. *Let $\Omega \subseteq E^n$ be an arbitrary unbounded open domain. Suppose that the coefficients of L satisfy*

$$a_{ij}(x, t)\beta_i\beta_j \geqslant 0 \quad \textit{for all } (x, t) \in \bar{\Omega} \times [0, T],\ \beta \in E^n,$$

and

$$|a_{ij}| \leqslant A(|x|^2+1)^{(2-\lambda)/2}, \qquad |b_i| \leqslant B(|x|^2+1)^{1/2}, \qquad c \leqslant C(|x|^2+1)^{\lambda/2}$$

in $\bar{\Omega} \times [0, T]$ for some positive constants A, B, and C. If u is a regular solution of $Lu \leqslant 0$ in $\Omega \times (0, T]$ such that

$$u(x, t) \geqslant 0 \quad \textit{for} \quad (x, t) \in \{\partial\Omega \times [0, T]\} \cup \{\Omega \times (t = 0)\}$$

and

$$u(x,t) \geqslant \begin{cases} -M\exp\{k\log(|x|^2+1)+1\}^2 & \text{if} \quad \lambda = 0, \\ -M\exp\{k(|x|^2+1)^{\lambda/2}\} & \text{if} \quad 0<\lambda\leqslant 2 \end{cases}$$

in $\Omega\times(0,T]$ *for some positive constants* M *and* k, *then* $u(x,t)\geqslant 0$ *in* $\bar{\Omega}\times[0,T]$.

Theorem B has been proved in the case $0<\lambda\leqslant 2$ by Bodanko in paper [5]. The proof for $\lambda = 0$ is given in the appendix to this note. Theorem B is, of course, also valid (with obvious changes) for the adjoint operator $\tilde{L}$ when condition (i) is satisfied.

The next theorem summarizes some of the results proved in [3]. In the case $\lambda = 0$ the results of paper [3] can be sharpened (cf. the appendix). However, the theorem which we quote below suffices for our purposes.

THEOREM AB1. *Suppose that the coefficients of* L *satisfy* (i) *and* (ii). *Then for some* $T_0 > 0$ *the fundamental solution* $\Gamma(x,t;\xi,\tau)$ *of* $Lu = 0$ *exists for all* $x, \xi\in E^n$, $0\leqslant\tau<t\leqslant T_0$ *and*

$$\int_{E^n}\Gamma(x,t,\xi,\tau)\,d\xi\leqslant Me^{G(x)}, \tag{2}$$

where

$$G(x) = \begin{cases} k\log(|x|^2+1) & \text{if} \quad \lambda = 0, \\ k(|x|^2+1)^{\lambda/2} & \text{if} \quad 0<\lambda\leqslant 2 \end{cases}$$

and M, k *are positive constants. Moreover,* $\Gamma(x,t;\xi,\tau)\geqslant 0$ *and it is the fundamental solution of the adjoint equation* $\tilde{L}u = 0$ *as a function of* (ξ,τ).

Note that we may assume that T in the definition of the strip S is $\leqslant T_0$. For if this is not the case, the proof of the uniqueness theorem can be carried out step by step in the usual way.

THEOREM AB2. *Suppose that the coefficients of* L *satisfy* (i) *and* (ii). *If* $u = u(x,t)$ *is a regular solution of* $Lu = 0$ *in* S *such that* $u(x,0) = 0$ *for* $x\in E^n$ *and*

$$\iint_S |u|\exp[-\{k_0\log(|x|^2+1)+1\}^2]\,dx\,dt<\infty \quad \text{if} \quad \lambda = 0$$

or

$$\iint_S |u|\exp\{-k_0(|x|^2+1)^{\lambda/2}\}\,dx\,dt<\infty \quad \text{if} \quad 0<\lambda\leqslant 2$$

for some positive constant k_0, *then* $u\equiv 0$ *in* $\bar{S}$.

Theorem AB2 is proved in paper [2] for $0<\lambda\leqslant 2$ and in the appendix to this note for $\lambda = 0$. We observe that both Theorems AB1 and AB2 hold if the coefficients of L satisfy (i) and if (ii) is replaced by the weaker condition (iii) $a_{ij}(x,t)\beta_i\beta_j \geqslant \gamma|\beta|^2$ for $(x,t)\in\bar{S}$ and $\beta\in E^n$. The role of condition (ii) will become apparent in the proof of Lemma 2 below.

Proof of the Theorem. The proof of our uniqueness theorem is essentially contained in the following three lemmas. The main idea is to show that every non-negative solution of $Lu = 0$ satisfies the growth condition of Theorem AB2.

LEMMA 1. *If the coefficients of L satisfy* (i) *and* (ii), *and if u is a non-negative regular solution of $Lu = 0$, then*

$$u(x,t) \geqslant \int_{E^n} \Gamma(x,t,\xi,\tau)u(\xi,\tau)d\xi$$

for all $x\in E^n$, $0\leqslant \tau < t \leqslant T$.

Proof. Let $h_\varrho = h_\varrho(x)$ be a continuous function defined on E^n such that $h_\varrho \equiv 1$ for $|x|\leqslant \varrho-1$, $h_\varrho\equiv 0$ for $|x|\geqslant\varrho$ and $0\leqslant h_\varrho\leqslant 1$, where $\varrho>1$ is arbitrary. We define

$$M_\varrho = \max_{\bar{S}} h_\varrho(x)u(x,t)$$

and consider the function

$$u_\varrho(x,t) = \int_{E^n} \Gamma(x,t;\xi,\tau)h_\varrho(\xi)u(\xi,\tau)d\xi$$

on $E^n\times(\tau,T]$. It follows from (1) that $u_\varrho(x,t)\leqslant M_\varrho e^{G(x)}$. By the definition of fundamental solution (cf. [3])

$$u_\varrho(x,t)\to h_\varrho(\bar{x})u(\bar{x},\tau)\leqslant u(\bar{x},\tau) \quad \text{as} \quad (x,t)\to(\bar{x},\tau+).$$

Moreover, $u_\varrho(x,t)$ is a regular solution of $Lu = 0$ in $E^n\times(\tau,T]$. Thus, by Theorem B (with $\Omega = E^n$), we obtain $u(x,t)-u_\varrho(x,t)\geqslant 0$. The assertion of the Lemma now follows from the Lebesgue monotone convergence theorem if we let $\varrho\to\infty$.

Our next result is an estimate from below for the fundamental solution of $Lu = 0$.

LEMMA 2. *Let the coefficients of L satisfy* (i) *and* (ii). *Fix $\bar{x}\in E^n$ and $t\in(0,T]$. Then for any $\varepsilon\in(0,t)$ there exist positive constants $\Lambda = \Lambda(\varepsilon,\bar{x})$, $\mu = \mu(\varepsilon,\bar{x})$ such that*

$$\text{(3)} \quad \Gamma(\bar{x},t;\xi,\tau)\geqslant \Lambda\begin{cases}\exp[-\mu\{\log(|\bar{x}-\xi|^2+1)\}^2] & \text{if} \quad \lambda = 0,\\ \exp(-\mu|\bar{x}-\xi|^\lambda) & \text{if} \quad 0<\lambda\leqslant 2\end{cases}$$

for all $(\xi,\tau)\in E^n\times[0,t-\varepsilon]$.

A similar result also holds with ξ fixed and x variable. A special case of Lemma 2 and the corresponding result with the roles of x and ξ

interchanged was proved in [4]. Note that it is here that we actually use condition (ii). If we assume that L satisfies (i) and (iii) then in place of (3) we would get (by essentially the same proof)

$$\Gamma(\bar{x}, t; \xi, \tau) \geqslant \Lambda \exp(-\mu|\bar{x}-\xi|^2) \tag{4}$$

regardless of the value of λ. If, for example, $L = \Delta - \partial/\partial t$, then clearly (i) and (iii) are satisfied and the fundamental solution has exactly the order of growth indicated by (4). Thus we cannot expect to get the estimate (3) without a condition stronger than (iii).

Proof. Suppose first that $0 < \lambda \leqslant 2$ and consider the function

$$V(\bar{x}, t; \xi, \tau) = \exp\left(\frac{-\nu|\bar{x}-\xi|^{\lambda}}{t-\tau-\varepsilon'}\right) \tag{5}$$

for $(\xi, \tau) \in D \equiv \{|\bar{x}-\xi| > R\} \times [0, t-\varepsilon')$, where $\varepsilon' = \varepsilon/2$, $R = |\bar{x}|+1$, and $\nu = \nu(\bar{x})$ is a positive constant which we shall choose so that $\tilde{L}V \geqslant 0$ for $(\xi, \tau) \in D$. Indeed we have

$$\begin{aligned}\tilde{L}V = \frac{V}{(t-\tau-\varepsilon')^2}\{&\nu^2\lambda^2|\bar{x}-\xi|^{2\lambda-4}a_{ij}(\bar{x}_i-\xi_i)(\bar{x}_j-\xi_j)-\\ &-\nu\lambda(\lambda-2)|\bar{x}-\xi|^{\lambda-4}(t-\tau-\varepsilon')a_{ij}(\bar{x}_i-\xi_i)(\bar{x}_j-\xi_j)-\\ &-\nu\lambda|\bar{x}-\xi|^{\lambda-2}(t-\tau-\varepsilon')a_{ii}+\nu\lambda|\bar{x}-\xi|^{\lambda-2}(t-\tau-\varepsilon')\tilde{b}_i(\bar{x}_i-\xi_i)+\\ &+\tilde{c}(t-\tau-\varepsilon')^2-\nu|\bar{x}-\xi|^{\lambda}\}.\end{aligned}$$

Thus in view of (1), (i) and (ii), $|\bar{x}-\xi| \geqslant R$, and $0 < \lambda \leqslant 2$ we obtain

$$\begin{aligned}\tilde{L}V \geqslant \frac{V}{(t-\tau-\varepsilon')^2}\{&\nu^2\lambda^2\gamma(|\xi|^2+1)^{(2-\lambda)/2}|\bar{x}-\xi|^{2\lambda-2}-\\ &-\nu\lambda nTK_1R^{-\lambda}(|\xi|^2+1)^{(2-\lambda)/2}|\bar{x}-\xi|^{2\lambda-2}-\\ &-\nu\lambda nT\tilde{K}_2(|\xi|^2+1)^{1/2}|\bar{x}-\xi|^{\lambda-1}-T^2\tilde{K}_3(|\xi|^2+1)^{\lambda/2}-\nu|\bar{x}-\xi|^{\lambda}\}\end{aligned} \tag{6}$$

for $(\xi, \tau) \in D$.

Set

$$K_0 = K_0(\bar{x}) = \inf_{\xi \in E^n} \frac{|\xi|^2+1}{|\bar{x}-\xi|^2}.$$

It is easy to show that $K_0 > 0$, whence

$$|\xi|^2+1 \geqslant K_0|\bar{x}-\xi|^2. \tag{7}$$

On the other hand, since $|\bar{x}-\xi| \geqslant R = |\bar{x}|+1$ in $\bar{D}$ we have

$$(|\xi|^2+1)^{\beta/2} \leqslant (|\xi|+1)^{\beta} \leqslant (|\xi-\bar{x}|+|\bar{x}|+1)^{\beta} \leqslant 2^{\beta}|\bar{x}-\xi|^{\beta} \tag{8}$$

for any $\beta > 0$. Making use of (7) and (8), it follows from (6) that

$$\tilde{L}V \geqslant \frac{|\overline{x}-\xi|^{\lambda}V}{(t-\tau-\varepsilon')^2}Q(\nu)$$

for $(\xi,\tau)\in D$, where

$$Q(\nu) = \lambda^2\gamma K_0^{(2-\lambda)/2}\nu^2-(2^{2-\lambda}n\lambda TK_1R^{-\lambda}+2n\lambda T\tilde{K}_2+1)\nu-2^{\lambda}T^2\tilde{K}_3.$$

Since $Q(0)<0$ it is clear that $Q(\nu)=0$ has at least one positive root. Thus if we set ν equal to the largest root of $Q(\nu)=0$ we have $\tilde{L}V\geqslant 0$ in D.

We define

$$\Lambda = \Lambda(\varepsilon,\overline{x}) = \min\Gamma(\overline{x},t;\xi,\tau) \quad \text{for} \quad (\xi,\tau)\in(|\overline{x}-\xi|\leqslant R)\times[0,t-\varepsilon'].$$

It follows from the strong maximum principle that $\Gamma(\overline{x},t;\xi,\tau)>0$ (cf. [4]). Thus $\Lambda>0$. Since Γ is the fundamental solution of $\tilde{L}u=0$ as a function of (ξ,τ), the function

$$z(\xi,\tau) = \Gamma(\overline{x},t;\xi,\tau)-\Lambda V(\overline{x},t;\xi,\tau)$$

satisfies $\tilde{L}z\leqslant 0$ in D. Moreover, $z\geqslant 0$ on the set $(|\overline{x}-\xi|=R)\times[0,t-\varepsilon']$ and $\lim z(\xi,\tau)\geqslant 0$ as $(\xi,\tau)\to(\zeta,t-\varepsilon')$ for any ζ such that $|\overline{x}-\zeta|\geqslant R$. Therefore, by Theorem B, we conclude that $z(\xi,\tau)\geqslant 0$ in $\overline{D}$ and, in particular, for $(\xi,\tau)\in(|\overline{x}-\xi|\geqslant R)\times[0,t-\varepsilon]$. In the latter set $t-\tau-\varepsilon'\geqslant\varepsilon'=\varepsilon/2$ so that we have

$$\Gamma(\overline{x},t;\xi,\tau)\geqslant\Lambda\exp(-\mu|\overline{x}-\xi|^{\lambda}), \tag{9}$$

where $\mu=2\nu/\varepsilon$. Finally, by the definition of Λ it follows that the estimate (9) holds throughout the strip $E^n\times[0,t-\varepsilon]$.

In case $\lambda=0$ the proof is quite similar except that the function V defined by (5) is replaced by

$$V(\overline{x},t;\xi,\tau) = \exp\left[-\frac{\nu}{t-\tau-\varepsilon'}\{\log(|\overline{x}-\xi|^2+1)\}^2\right].$$

Proceeding as in the case $\lambda>0$ and using formula (8) together with $\log(|\overline{x}-\xi|^2+1)\geqslant\log 2$ for $|\overline{x}-\xi|\geqslant R$ and

$$\inf_{|\overline{x}-\xi|\geqslant 1}\frac{(|\xi|^2+1)|\overline{x}-\xi|^2}{(|\overline{x}-\xi|^2+1)^2}>0$$

we obtain the estimate

$$\tilde{L}V\geqslant\frac{\{\log(|\overline{x}-\xi|^2+1)\}^2}{(t-\tau-\varepsilon')^2}VQ^*(\nu),$$

where again $Q^*(\nu)$ is a quadratic polynomial in ν such that $Q^*(0)<0$ and $Q^*(\nu)\to+\infty$ as $\nu\to+\infty$.

We now apply Lemmas 1 and 2 to prove

LEMMA 3. *Suppose u is a non-negative regular solution of $Lu = 0$ in S. Then*

$$(10)\qquad \int_0^{T/2}\int_{E^n} u(\xi,\tau)\exp[-\mu\{\log(|\xi|^2+1)\}^2]d\xi d\tau < \infty \quad \text{if} \quad \lambda = 0$$

or

$$(11)\qquad \int_0^{T/2}\int_{E^n} u(\xi,\tau)\exp(-\mu|\xi|^\lambda)d\xi d\tau < \infty \quad \text{if} \quad 0<\lambda\leqslant 2,$$

where $\mu = \mu(T/2, 0)$.

Proof. Apply Lemma 1 with $\bar{x} = 0$ and $t = T$ to obtain

$$\int_{E^n} \Gamma(0, T; \xi, \tau)u(\xi,\tau)d\xi \leqslant u(0, T)$$

for $\tau \in [0, T)$. Hence, by Lemma 2 with $\bar{x} = 0$, $t = T$ and $\varepsilon = T/2$ we have

$$A\int_{E^n} u(\xi,\tau)\exp[-\mu\{\log(|\xi|^2+1)\}^2]d\xi \leqslant u(0,T) \quad \text{if} \quad \lambda = 0$$

or

$$A\int_{E^n} u(\xi,\tau)\exp(-\mu|\xi|^\lambda)d\xi \leqslant u(0,T) \quad \text{if} \quad 0<\lambda\leqslant 2$$

for $\tau \in [0, T/2]$, where $\mu = \mu(T/2, 0)$. The assertion of the Lemma now follows by integrating these expressions with respect to τ on $[0, T/2]$.

Our uniqueness theorem is an easy consequence of Lemma 3. Since u_1 and u_2 are non-negative regular solutions of $Lu = 0$ in S each of them must satisfy the growth condition (10) or (11). Let $v = u_1 - u_2$. Then v is a regular solution of the Cauchy problem $Lv = 0$ in S, $v(x, 0) = 0$ for all $x \in E^n$. Moreover, $|v| \leqslant u_1 + u_2$ implies that v also satisfies the growth condition (10) or (11). It follows from Theorem AB2 that $v \equiv 0$ in $E^n \times [0, T/2]$. The proof of the Theorem is now completed by applying the same argument in the strip $E^n \times [T/2, T]$.

APPENDIX

We shall now prove the assertions of Theorems B and AB2 for $\lambda = 0$. We shall also indicate how the results of paper [3] can be sharpened in the case $\lambda = 0$.

1. Proof of Theorem B. For arbitrary $\varrho > 0$ and $\beta > 0$ consider the function

$$w(x,t) = u(x,t) + M\exp[2e^{\beta t}\{k\log(r^2+1)+1\}^2 - \{k\log(\varrho^2+1)+1\}^2],$$

where $r = |x|$. It is clear that $w(x,t) \geqslant 0$ for $(x,t) \in \{\partial\Omega \times [0,T]\} \cup \{\Omega \times (t=0)\}$ and for $(x,t) \in \{\Omega \times [0,T]\} \cap \{(|x| = \varrho) \times [0,T]\}$. On the other hand, it is easily verified that if $t \in (0, 1/\beta]$ we have

$$LW \leqslant e^{\beta t} W \{k \log(r^2+1)+1\}^2 (E-2\beta),$$

where $W = \exp[2e^{\beta t}\{k\log(r^2+1)+1\}^2]$ and E is a positive constant which depends only on A, B, C, k and n. Thus if we set $\beta = E$ it follows that $Lw < 0$ in $\Omega \times (0, 1/\beta]$. Now let (ξ, τ) be any fixed point in $\Omega \times (0, 1/\beta]$ and assume that $\varrho > |\xi|$. By the weak maximum principle applied to w in $\{\Omega \times (0, 1/\beta]\} \cap \{(|x| < \varrho) \times (0, 1/\beta]\}$, $w(\xi, \tau) \geqslant 0$. Therefore if we let $\varrho \to +\infty$ we obtain $u(\xi, \tau) \geqslant 0$ in the strip $\bar{\Omega} \times [0, 1/\beta]$. The proof of Theorem B for $t > 1/\beta$ can now be carried out in the standard manner.

2. Proof of Theorem AB2. Theorem AB2 is consequence of the following result which is proved in [2]. We consider the operator

$$\mathscr{L}u \equiv \{A_{ij}(x,t)u\}_{x_i x_j} - \{A_i(x,t)u\}_{x_i} + A(x,t)u - u_t \tag{12}$$

for $(x,t) \in S$. Let $S_\delta = E^n \times (0, \delta]$ and $\bar{S}_\delta = E^n \times [0, \delta]$ for any $\delta \in (0, T]$.

THEOREM AB2*. *Let u be such that $\mathscr{L}u = 0$ in S and $u(x,0) = 0$ and let δ be a fixed number in $(0,T]$. If there exists a function $\varphi = \varphi(x,t)$ such that $\varphi \in C^2(\bar{S}_\delta)$, $\varphi > 0$ in every compact subset of $\bar{S}_\delta$,*

$$\mathscr{L}^*\varphi \equiv A_{ij}(x,t)\varphi_{x_i x_j} + A_i(x,t)\varphi_{x_i} + A(x,t)\varphi + \varphi_t \leqslant 0 \tag{13}$$

in $\bar{S}_\delta$ and

$$\iint\limits_{S_\delta} |u| \Big\{\max_i \sum_j |A_{ij}\varphi_{x_j}| + \varphi(\max_{i,j}|A_{ij}| + \max_i |A_i|)\Big\} dx\,dt < \infty \tag{14}$$

then $u \equiv 0$ in S.

In view of (1) for $\lambda = 0$ it is clear that we can write L in the form (12) where $A_{ij}\beta_i\beta_j \geqslant 0$ and

$$|A_{ij}| \leqslant M_1(|x|^2+1), \qquad |A_i| \leqslant M_2(|x|^2+1)^{1/2}, \qquad A \leqslant M_3$$

for some positive constants M_1, M_2, M_3. Set

$$\varphi(x,t) = \exp\left[\frac{-1}{1-\beta t}\{k\log(|x|^2+1)+2\}^2\right]$$

for $(x,t) \in E^n \times [0, 1/2\beta]$, where $k = k_0 + 1$ and $\beta > 0$ is a constant to be specified below. It is easily verified that

$$\mathscr{L}^*\varphi \leqslant \frac{\varphi}{(1-\beta t)^2}\{k\log(|x|^2+1)+2\}^2(E-\beta),$$

where E is a positive constant which depends only on M_1, M_2, M_3, k and n. Thus, if we set $\beta = 2E$, condition (13) is satisfied in $E^n \times [0, 1/2\beta]$.

To prove Theorem AB2 we must show that

$$\iint_S |u| \exp[-\{k_0 \log(|x|^2+1)+1\}^2]\,dx\,dt < \infty \tag{15}$$

implies that (14) is satisfied. It is not difficult to show that

$$\max_i \sum_j |A_{ij}\varphi_{x_i}| + \varphi(\max_{i,j}|A_{ij}| + \max_i |A_i|) \leqslant (|x|^2+1)F\varphi,$$

where F is a constant which depends only on M_1, M_2, k and n. Since $0 \leqslant t \leqslant 1/2\beta$ and $k = k_0+1$, we have

$$\varphi \leqslant \exp[-\{k\log(|x|^2+1)+2\}^2] \leqslant (|x|^2+1)^{-1}\exp[-\{k_0\log(|x|^2+1)+1\}^2]$$

and hence condition (14) is a consequence of (15).

Theorem AB2 corresponds to Theorem 1 of [2]. By an argument similar to the one given above Theorems 3 and 4 of [2] can also be sharpened in the case $\lambda = 0$. We omit the details.

3. Remarks concerning paper [3]. In paper [3] we prove the existence of a fundamental solution $\Gamma(x, t; \xi, \tau)$ of the equation $\mathscr{L}u = 0$, where

$$\mathscr{L}u \equiv u_t - \{a_{ij}(x, t)u_{x_i} + a_j(x, t)u\}_{x_j} - b_j(x, t)u_{x_j} - c(x, t)u.$$

We assume that the coefficients of $\mathscr{L}$ satisfy

$$\nu|\beta|^2 \leqslant a_{ij}(x, t)\beta_i\beta_j \leqslant k_1(|x|^2+1)^{(2-\lambda)/2}|\beta|^2$$

and

$$|a_{ij,x_j}|,\ |a_j|,\ |b_j| \leqslant k_2(|x|^2+1)^{1/2}, \qquad c+a_{j,j},\ c-b_{j,j} \leqslant k_3(|x|^2+1)^{\lambda/2}.$$

In particular, we prove that for arbitrary $a > 0$ there exists a $T_a > 0$ such that

$$\Gamma(x, t; \xi, \tau) = \gamma_a(x, t; \xi, \tau)\exp\{g_a(x, t) - g_a(\xi, \tau)\}$$

for all $x, \xi \in E^n$ and $0 \leqslant \tau < t \leqslant T_a$, where γ_a is the fundamental solution of an equation related to $\mathscr{L}u = 0$,

$$\int_{E^n} \gamma_a(x, t; \xi, \tau)\,d\xi \leqslant 1 \quad \text{for all } (x, t) \in E^n \times (\tau, T_a]$$

and g_a is a function which satisfies

$$C^a \equiv k_1(|x|^2+1)^{(2-\lambda)/2}|\nabla_x g_a|^2 + (2+n)k_2(|x|^2+1)^{1/2}|\nabla_x g_a| + k_3(|x|^2+1)^{\lambda/2} + |a_{ij}g_{ax_ix_j}| - g_{at} < 0 \tag{16}$$

for all $(x, t) \in E^n \times [0, T_a]$. For proving the existence of the fundamental solution Γ it makes no difference how g_a is chosen as long as (16) is satisfied. In paper [3] we also apply the fundamental solution to derive a representation formula for the solution of the Cauchy problem for $\mathscr{L}$. In this application the choice of g_a is important since it determines the admissible growth classes for the data. For the case $\lambda = 0$ in [3] we used the function

$$g_a(x,t) = \log \frac{(|x|^2+1)^a}{1-\beta(a)t},$$

where $\beta(a) > 0$ is a function of a which is determined so that (16) holds. With this choice of g_a we were able to solve the Cauchy problem for data which grows no faster than $\mathrm{const}(|x|^2+1)^a$.

We observe that the function

$$g_a(x,t) = \frac{1}{1-\beta t}\{a\log(|x|^2+1)+1\}^2$$

is a better choice for g_a. Indeed, if $t \in [0, 1/2\beta]$ we have

$$C^a \leqslant \frac{\{a\log(|x|^2+1)+1\}^2}{(1-\beta t)^2}(E-\beta),$$

where E is a positive constant which depends only on a, k_1, k_2, k_3 and n. Thus if we take $\beta(a) = 2E$ and $T_a = 1/2\beta$, then (16) holds. With this choice of g_a the representation formula for the solution of the Cauchy problem given in Theorem III of [3] is valid in the case $\lambda = 0$ for data which grows no faster than

$$\mathrm{const}\exp\{a\log(|x|^2+1)+1\}^2.$$

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