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ON A CERTAIN PROPERTY OF A TRANSFORMATION IN HILBERT SPACE IN CONNECTION WITH THE THEORY OF DIFFERENTIAL EQUATIONS

BY

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In the proof by the method of successive approximations of the theorem on the existence of a solution of the differential equation

$$y' = f(x, y),$$

where $y = (y_1, \ldots, y_n)$ and $f = (f_1, \ldots, f_n)$, it is necessary to distinguish two different cases.

1° If the function f satisfies the Lipschitz condition with a certain constant N in the whole space, then the proof is almost automatical.

2° In the case where the function f is defined only in a certain closed set Z, the proof of the local existence theorem involves some slight but troublesome complications.

One way of reducing case 2° to case 1° is to extend the function f from the set Z to the whole space in such a way that the Lipschitz condition be satisfied. Such continuation of the function f may be achieved in various ways. If the set Z is the Cartesian product of a segment and a convex set W, then the function f may be extended in a rather simple manner to the whole space. The subsequent construction seems to be of interest, because it may be applied also in case of a Hilbert space.

The principal step is the definition of a contraction T which maps the whole space H onto a convex and closed set W (see Theorem 1).

The present paper is an answer to the problem set by T. Ważewski.

1. Let W be a convex closed set and let $W \subset H$, where H is a Hilbert space. For an arbitrary point $P \in H$ there exists [1] precisely one point $Q \in W$ such that

$$r(P, W) = r(P, Q) \ [2]. \tag{1}$$

[1] See e.g. N. Bourbaki, *Espaces vectoriels topologiques*, Paris 1955.

[2] $r(P, W)$ and $r(P, Q)$ are distances of the point P from the set W and from the point Q respectively, and

$$r^2(P, Q) = (Q-P,\ Q-P) = \|Q-P\|^2.$$

We define the transformation T by setting

$$T(P) = Q. \tag{2}$$

This transformation T has the following properties:

1° *$T(P) = P$ if $P \in W$;*

2° *$T(P) \in W$ for $P \in H$;*

3° *$r(P, W) = r\big(P, T(P)\big)$ for $P \in H$;*

4° *for arbitrary $A, B \in H$*

$$r\big(A, [T(A), T(B)]\big) = r\big(A, T(A)\big),$$

where $[T(A), T(B)]$ is the closed segment joining $T(A)$ and $T(B)$.

Properties 1°-3° follow from the definition of the transformation T. Therefore, let us prove property 4°.

In view of the convexity of the set W, we have

$$T(A) \in [T(A), T(B)] \subset W,$$

whence

$$r(A, W) \leqslant r\big(A, [T(A), T(B)]\big) \leqslant r\big(A, T(A)\big).$$

Hence property 4° follows.

FUNDAMENTAL THEOREM. *Transformation T is a contraction, i.e.*

$$r\big(T(A), T(B)\big) \leqslant r(A, B) \tag{3}$$

for arbitrary points $A, B \in H$.

Proof. First let us prove for scalar products two following inequalities:

$$\operatorname{Re}\big(T(B) - T(A), A - T(A)\big) \leqslant 0, \tag{4}$$

$$\operatorname{Re}\big(T(A) - T(B), B - T(B)\big) \leqslant 0. \tag{5}$$

Writing

$$P(\varrho) = T(A) + \varrho\big(T(B) - T(A)\big), \tag{6}$$

where ϱ is a real parameter and $0 \leqslant \varrho \leqslant 1$, we can get for property 4° the form

$$\|A - T(A)\|^2 \leqslant \|A - P(\varrho)\|^2 \tag{7}$$

for $0 \leqslant \varrho \leqslant 1$.

Substituting (6) in (7) we obtain

$$\begin{aligned}\|A - T(A)\|^2 \leqslant \Big\|A - \big(T(A) - \varrho\big(T(B) - T(A)\big)\big)\Big\|^2 \\ = \|A - T(A)\|^2 - 2\varrho \operatorname{Re}\big(A - T(A), T(B) - T(A)\big) + \\ + \varrho^2 \|T(B) - T(A)\|^2.\end{aligned}$$

Hence

$$2\varrho\,\mathrm{Re}\big(A-T(A),\,T(B)-T(A)\big)\leqslant \varrho^2\|T(B)-T(A)\|^2.$$

Dividing by ϱ (under the assumption that $\varrho\neq 0$, i.e. $0<\varrho\leqslant 1$) and letting it tend to 0 ($\varrho\to 0+$), we obtain inequality (4). Inequality (5) follows from (4) by interchanging A and B.

Let us now introduce the auxiliary points

$$A^*(\varrho)=T(A)+\varrho\big(A-T(A)\big),\qquad B^*(\varrho)=T(B)+\varrho\big(B-T(B)\big)$$

for $0\leqslant\varrho\leqslant 1$ and the function

$$h(\varrho)=\|B^*(\varrho)-A^*(\varrho)\|^2.$$

From the definition of $A^*(\varrho)$ and $B^*(\varrho)$ we have

$$h(0)=\|T(B)-T(A)\|^2,\qquad h(1)=\|B-A\|^2. \tag{8}$$

To prove the theorem it suffices to show that $h(0)\leqslant h(1)$.

Computing the derivatives $h'(\varrho)$ and $h''(\varrho)$ we infer that

$$h'(\varrho)=2\mathrm{Re}\Big(B^*(\varrho)-A^*(\varrho),\,\big\{\big(B-T(B)\big)-\big(A-T(A)\big)\big\}\Big),$$

$$h''(\varrho)=2\big\|B-T(B)-\big(A-T(A)\big)\big\|^2=\text{const}\geqslant 0.$$

It follows that $h'(\varrho)$ is non-decreasing. In view of (4) and (5) we see that $h'(0)\geqslant 0$, hence

$$h'(\varrho)\geqslant 0\qquad\text{for}\qquad 0\leqslant\varrho\leqslant 1.$$

This implies that the function $h(\varrho)$ is non-decreasing. Therefore $h(0)\leqslant h(1)$ which means that (in view of (8)) inequality (3) holds.

2. Let the function $\varphi(P)$ be defined in the convex and closed set $W\subset H$ and nowhere else. By means of the function φ we define the new function φ^* by putting

$$\varphi^*(A)=\varphi[T(A)]\qquad\text{for}\qquad A\in H.$$

From 1° it follows that for an arbitrary point $B\in W$ we have $\varphi^*(B)\equiv\varphi(B)$; therefore the function φ^* is a continuation of the function φ to the whole space H.

From the above definition of the function φ^* it results that 1° the sets of values of functions φ and φ^* are identical; 2° if the function φ satisfies in W condition

(S) for arbitrary points $B_1, B_2\in W$ we have

$$|\varphi(B_1)-\varphi(B_2)|\leqslant\Omega\big(r(B_1,B_2)\big),$$

where the function $\Omega(t)$ is non-negative and non-decreasing for $t\in[0,+\infty)$,

then the function φ^* will also fulfil this condition in H.

Indeed, in view of the Fundamental Theorem we have the relation

$$|\varphi^*(A_1)-\varphi^*(A_2)| = |\varphi[T(A_1)]-\varphi[T(A_2)]$$

$$\leqslant \Omega\big(r\big(T(A_1), T(A_2)\big)\big) \leqslant \Omega\big(r(A_1, A_2)\big)$$

for arbitrary points $A_1, A_2 \epsilon H$.

Assuming in particular that $\Omega(t) = Nt^a$, where $N > 0$ and $a > 0$, we obtain from condition (S) the Hölder condition.

3. If the function $\psi(P_1, \ldots, P_n)$ is defined only for $P_i \epsilon W_i \subset H_i$ $(i = 1, \ldots, n)$, where W_i $(i = 1, \ldots, n)$ are convex and closed sets contained in Hilbert spaces H_i $(i = 1, \ldots, n)$ respectively, then the function ψ may be extended to the whole space $H_1 \times \ldots \times H_n$ (or to the set $W_1 \times \ldots \times W_0 \times H_{s+1} \times \ldots \times H_n$, $1 \leqslant s \leqslant n-1$) in such a way that certain properties (analogous to those of the function φ) be preserved.

For that purpose we define the transformation T_i $(i = 1, \ldots, n)$ in each space H_i separately analagously to (1) so that

$$T_i(H_i) = W_i \quad (i = 1, \ldots, n).$$

The function

$$\psi^*(A_1, \ldots, A_n) = \psi\big(T_1(A_1), \ldots, T_n(A_n)\big),$$

where $(A_1, \ldots, A_n) \epsilon H_1 \times \ldots \times H_n$ (or $(A_1, \ldots, A_s, A_{s+1}, \ldots, A_n) \epsilon W_1 \times \ldots \times W_s \times H_{s+1} \times \ldots \times H_s$) is the required continuation of the function ψ to the whole space $H_1 \times \ldots \times H_n$ (or to the set $W_1 \times \ldots \times W_s \times H_{s+1} \times \ldots \times H_n$).

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