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ON STRONGLY ADDITIVE SET FUNCTIONS

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The purpose of this note is to give another proof of a theorem of J. Kisyński on strongly additive set functions ([1]).

Let $\mathscr{L}$ be a lattice of sets, i.e. a family of sets closed under joints and meets. Let λ be a function from $\mathscr{L}$ into an Abelian group G such that

$$\lambda(A)+\lambda(B) = \lambda(A \cup B)+\lambda(A \cap B) \quad \text{for } A, B \in \mathscr{L}. \tag{i}$$

Hence, by induction,

$$\lambda(A_1)+\ldots+\lambda(A_n) = \sum_{k=1}^{n} \lambda\Big(\bigcup_{1\leqslant i_1<\ldots<i_k\leqslant n} A_{i_1} \cap \ldots \cap A_{i_k}\Big). \tag{1}$$

In fact, by (i) we have

$$\begin{aligned} \lambda\Big(\bigcup_{1\leqslant i_1<\ldots<i_{l+1}\leqslant n} A_{i_1} \cap \ldots \cap A_{i_{l+1}}\Big)+\lambda\Big(\bigcup_{1\leqslant j_1<\ldots<j_l\leqslant n} A_{j_1} \cap \ldots \cap A_{j_l} \cap A_{n+1}\Big) \\ = \lambda\Big(\bigcup_{1\leqslant i_1<\ldots<i_{l+1}\leqslant n+1} A_{i_1} \cap \ldots \cap A_{i_{l+1}}\Big) + \\ +\lambda\Big(\bigcup_{1\leqslant j_1<\ldots<j_{l+1}\leqslant n} A_{j_1} \cap \ldots \cap A_{j_{l+1}} \cap A_{n+1}\Big). \end{aligned} \tag{2}$$

Summing up equations (2) for $l = 0, 1, \ldots, n-1$ and applying inductive hypothesis we obtain (1) for $A_1, \ldots, A_{n+1} \in \mathscr{L}$.

Now we are going to prove

LEMMA. *If $A_1, \ldots, A_n, B_1, \ldots, B_n$ are in $\mathscr{L}$ and*

$$\chi_{A_1}+\ldots+\chi_{A_n} = \chi_{B_1}+\ldots+\chi_{B_n}, \tag{3}$$

then

$$\lambda(A_1)+\ldots+\lambda(A_n) = \lambda(B_1)+\ldots+\lambda(B_n). \tag{4}$$

([1]) J. Kisyński, *Remark on strongly additive set functions*, Fundamenta Mathematicae 63 (1968), p. 327–332.

Proof. Since $\chi_{A_1}(p)+\ldots+\chi_{A_n}(p)$ is the number of the A's containing the point p, formula (3) implies

$$\bigcup_{1\leqslant i_1<\ldots<i_k\leqslant n} A_{i_1}\cap\ldots\cap A_{i_k} = \bigcup_{1\leqslant j_1<\ldots<j_k\leqslant n} B_{j_1}\cap\ldots\cap B_{j_k},$$

whence in view of (1) we get (4).

From now on we assume that the empty set $\emptyset$ is in $\mathscr{L}$ and that

(ii) $$\lambda(\emptyset) = 0.$$

A function λ on $\mathscr{L}$ satisfying (i) and (ii) is called *strongly additive.*

THEOREM. *Every strongly additive set function λ on $\mathscr{L}$ has the unique extension to an additive set function defined on the ring generated by $\mathscr{L}$.*

Proof. Consider the set P of all functions f of the form $f = \sum_{i=1}^{n}\chi_{A_i}$, where $A_i \in \mathscr{L}$, and put $I(f) = \sum_{i=1}^{n}\lambda(A_i)$. P is an Abelian semigroup. The functional I is well defined (see the lemma and condition (ii)) and additive on P. Moreover, it can be uniquely extended to an additive functional $I^*: P^*\to G$, where $P^* = \{f-g: f, g\in P\}$, by $I^*(f-g) = I(f)-I(g)$. Consider the family of sets $\mathfrak{R} = \{A: \chi_A\in P^*\}$. Clearly, $\mathscr{L}\subset\mathfrak{R}$. Since $\chi_{A_1\cap A_2} = \chi_{A_1}\cdot\chi_{A_2}$, $\chi_{A_1\setminus A_2} = \chi_{A_1}-\chi_{A_1\cap A_2}$ and $\chi_{A_1\cup A_2} = \chi_{A_1}+\chi_{A_2\setminus A_1}$, $\mathfrak{R}$ is a ring. Putting $\mu(A) = I^*(\chi_A)$ for $A\in\mathfrak{R}$, we obtain the required extension of λ. The uniqueness of μ is a consequence of the construction.

Remark (added in proof). I have been recently informed that the above theorem was also demonstrated in the paper *On the extension of measures* by B. J. Pettis, Annals of Mathematics, 54 (1951), p. 186–197 (p. 188, Theorem 1.2). The idea of the proof, which slightly differs from that of Kisyński, is based on the notion of a semiring.

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