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ON SOME INTEGRAL INEQUALITIES INVOLVING CHEBYSHEV WEIGHT FUNCTION

BY

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Let a real function $f = f(t)$, defined on the interval $[-1, 1]$ and absolutely continuous there, satisfy the boundary conditions

$$f(-1) = f(1) = 0. \tag{1}$$

We shall denote the class of all such functions by Φ. Consider the inequality

$$\lambda \int p^b f^2 \leqslant \int p^a \dot{f}^2, \tag{2}$$

where $\int$ denotes $\int_{-1}^{1} dt$, $p = (1-t^2)^{-1/2}$, $f \in \Phi$, $\dot{f} = df/dt$ and a, b and λ are some constants. Assume that inequality (2) holds true for any $f \in \Phi$ (as, e.g., for $\lambda = 0$). We are interested, for given a and b, in the best possible value of λ in (2). Such λ will be denoted by $\lambda(a, b)$.

The values of $\lambda(a, b)$ are well known in cases $a = b = 0$, $a = 0$, $b = 2$ (see [1], Theorems 257 and 262) and $a = 0$, $b = 4$ ([2], formula 1,12). If conditions (1) are replaced either by the condition $\int f = 0$ or by the conditions $\int f = 0$ and $f(-1) = f(1)$, then the corresponding best values of λ in (2) are also known for a number of pairs a and b (see [1], Theorems 225 and 258, [2], Theorem 1.3*). The inequalities like (2) with $a = 0$ but arbitrary function $p(t)$ have been derived in [2].

The inequalities of the form (2) are useful, among others, in some problems of approximation for ordinary differential equations (see [3] and [4]). In the note [5], the second author was forced to consider inequality (2) for $a = 1$ and $b = 5$ and he established there that $1/0.571 < \lambda(1,5)$. Looking for the exact value of $\lambda(1,5)$, which is $1/2$, we have considered more general cases $b = a+4$ and $b = a+2$.

THEOREM. *For any $f \in \Phi$ and $a > -1$, the following inequality holds true:*

$$(1+a) \int p^{a+4} f^2 \leqslant \int p^a \dot{f}^2. \tag{3}$$

There is equality in (3) *only if either* $a > 0$ *and* $f = \mathrm{const}(1-t^2)^{(1+a)/2}$ *or* $a \leqslant 0$ *and* $f = 0$ *besides the trivial case when the integrals in* (3) *become infinite.*

We also have, for any $f \in \Phi$ *and* $a > -2$, *the inequality*

$$(2+a)\int p^{a+2}f^2 \leqslant \int p^a \dot{f}^2. \tag{4}$$

There is equality in (4) *only if* $f = \mathrm{const}(1-t^2)^{1+a/2}$ *besides the trivial case as above.*

From this theorem we easily deduce the following

COROLLARY. *We have* $\lambda(a, a+4) = 1+a$ *for* $a > 0$ *and* $\lambda(a, a+4) \geqslant 1+a$ *for* $-1 < a \leqslant 0$. *We also have* $\lambda(a, a+2) = a+2$ *for* $a > -2$.

In the proof of the Theorem the method of integral identities will be used (see [6], Ch. V, § 13).

We can assume, without loss of generality, that

$$\int p^a \dot{f}^2 < \infty \quad \text{and} \quad a > -2. \tag{5}$$

We shall prove that this assumption implies the relations

$$p^{a+2}f^2 \to 0 \quad \text{for} \quad t \to \pm 1. \tag{6}$$

To do this, note that due to the boundary condition $f(1) = 0$ we have

$$f^2(t) = \Big(\int_t^1 \dot{f}\Big)^2 \leqslant \int_t^1 p^a \dot{f}^2 \int_t^1 p^{-a},$$

and, moreover,

$$\int_t^1 p^{-a} \leqslant 2(a+2)^{-1}p^{-2-a}(t),$$

if only $a > -2$ and $t \geqslant 1/2$. Hence, in view of (5), we obtain $p^{a+2}f^2 \to 0$ for $t \to 1$. A similar procedure applies also to the case $t \to -1$.

To get a suitable integral identity, let us put

$$F = p^a \dot{f}^2 - \lambda p^b f^2 \quad \text{and} \quad f = p^c h, \tag{7}$$

where a, b, c and λ are constants and $h = h(t)$ is a new function. Expressing F in terms of h, we get

$$F = p^{2c+a}\dot{h}^2 + 2vh\dot{h} + wh^2,$$

where $v = ctp^{a+2c+2}$ and $w = c^2t^2p^{a+2c+4} - \lambda p^{b+2c}$. We demand now that $\dot{v} = w$. It will be so only if the constants a, b, c, λ are connected either by the relations

$$b = a+4, \quad c = -1-a, \quad \lambda = 1+a \tag{8}$$

or by the relations

$$b = a+2, \quad c = -2-a, \quad \lambda = 2+a. \tag{9}$$

There is one case more for which $\dot{v} = w$, namely $\lambda = c = 0$ and a, b arbitrary. However, this last case does not give rise to any interesting identity.

We get, for both cases (8) and (9), the formula

$$F = p^{2c+a}\dot{h}^2 + d/dt(ctp^{a+2}f^2). \tag{10}$$

In view of (5), (6), (7) and (10) we obtain the desired integral identity

$$\int p^a \dot{f}^2 - \lambda \int p^b f^2 = \int p^{2c+a}\dot{h}^2, \tag{11}$$

in which the right-hand member is, evidently, non-negative. Taking into account (8) and (9) resp., we get from (11) inequalities (3) and (4) resp.

Now the equality in (3) and in (4) may occur only if $h = C = \text{const}$, which gives

$$f = Cp^c, \tag{12}$$

and condition (5) for this f takes the form $C^2 \int p^{2c+a+4} < \infty$. This last condition is satisfied for every C if

$$2c + a + 2 < 0. \tag{13}$$

If, however, $2c + a + 2 \geqslant 0$, then C must be zero.

Consider separately the cases $c = -2 - a$, $a > -2$, and $c = -1 - a$, $a > -1$. In the first case the restriction (13) is satisfied, hence, for $a > -2$, all functions (12), which take the form $Cp^{(1+a)/2}$, realize the equality in (4). In the second case the restriction (13) is satisfied only for $a > 0$. Hence, for $a > 0$, all functions (12), which are in this case of the form $Cp^{1+a/2}$, turn (3) into equality. For $-1 < a \leqslant 0$, (13) is not satisfied, hence $f = 0 \cdot p^c = 0$. This completes the proof of the theorem.

Remark. Inequalities (3) and (4) are valid under a little milder assumptions. It suffices to suppose that

(i) the function f is absolutely continuous in every closed subinterval of the open interval $(-1, 1)$;

(ii) both limits $\lim_{t\to+1} f(t)$ and $\lim_{t\to-1} f(t)$ are equal to zero if they exist.

It is easily seen that (i) and (5) imply both the existence of $\lim_{t\to+1} f(t)$ and of $\lim_{t\to-1} f(t)$.

Let us return to $\lambda(a, b)$. Note that the inequality

$$\lambda(a, b_1)\lambda(a, b_2) \leqslant \lambda^2\left(a, \frac{b_1 + b_2}{2}\right) \tag{14}$$

holds true. In fact, we have

$$\lambda(a, b_1)\lambda(a, b_2)\int p^{b_1}f^2 \int p^{b_2}f^2 \leqslant \left(\int p^a \dot{f}^2\right)^2,$$

whence, by Schwarz inequality, we get

$$\lambda(a, b_1)\lambda(a, b_2)\Big(\int p^{(b_1+b_2)/2} f^2\Big)^2 \leqslant \Big(\int p^a \dot{f}^2\Big)^2,$$

from which (14) follows at once.

Apply inequality (14) to the particular case $a = 0$, $b_1 = 0$, $b_2 = 2$. Taking into account the known values $\lambda(0, 0) = (\pi/2)^2$ and $\lambda(0, 2) = 2$, we get the estimate $\lambda(0, 1) \geqslant \pi/\sqrt{2}$. On the other hand, we have for the function $f = 1 - t^2$ the equality $\int \dot{f}^2 / \int pf^2 = 64\pi/9$, hence $\lambda(0, 1) \leqslant 64\pi/9$. So we get $2.22 < \lambda(0, 1) < 2.27$, a result which is useful in estimating some integrals.

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