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T-INDEPENDENCE IN DISTRIBUTIVE LATTICES

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Let $(X;\ \mathscr{F})$ be a universal algebra. We say that the set $I \subseteq X$ is *T-independent* if for any non-trivial algebraic operation $f(x_1, \ldots, x_n)$ and any sequence of different elements $a_1, \ldots, a_n \in I$ we have $f(a_1, \ldots, a_n) \neq a_i$ for $i = 1, \ldots, n$ (cf. [1]). By *M-independence* we mean the usual algebraic independence in the sense of Marczewski [2]; *C-independence* in an algebra $(X;\ \mathscr{F})$ is closure independence with respect to the closure system of subalgebras of $(X;\ \mathscr{F})$.

It is easy to show that

(i) M-independence implies T-independence which implies C-independence;

(ii) I is T-independent iff $f(a_1, \ldots, a_n) \notin I$ for any non-trivial algebraic operation $f(x_1, \ldots, x_n)$ and different $a_1, \ldots, a_n \in I$.

In this paper we consider T-independence in distributive lattices and also a more general notion of independence in distributive lattices which contains M-independence and T-independence as special cases.

Marczewski proved (see [3]) that

(iii) the set $I \subseteq X$ is M-independent in a distributive lattice $(X;\ +, \cdot)$ iff, for any different $a_1, \ldots, a_n \in I$ and any k $(1 \leqslant k \leqslant n-1)$,

$$\prod_{1 \leqslant i \leqslant k} a_i \nleqslant \sum_{k+1 \leqslant j \leqslant n} a_j.$$

THEOREM 1. *The set $I \subseteq X$ is T-independent in a distributive lattice $(X;\ +, \cdot)$ iff, for any different elements $a_1, \ldots, a_n \in I$,*

$$a_1 \nleqslant \sum_{1 < i \leqslant n} a_i \quad \text{and} \quad \prod_{1 < i \leqslant n} a_i \nleqslant a_1.$$

Remark. Note that this means that I is T-independent in a distributive lattice $(X;\ +, \cdot)$ iff it is M-independent in the semilattices $(X;\ +)$ and $(X;\ \cdot)$ (cf. Szász [4]).

Proof of Theorem 1. Suppose that

$$a_1 \leqslant \sum_{1 < i \leqslant n} a_i;$$

we have

$$a_1\cdot\sum_{1<i\leqslant n} a_i = a_1.$$

But the operation $x_1\cdot(x_2+\ldots+x_n)$ is not trivial, so the set I is dependent.

Suppose that the set I is dependent. Then there exist a non-trivial operation $f(x_1,\ldots,x_n)$ and elements $a_1,\ldots,a_n\in I$ such that $f(a_1,\ldots,a_n) = a_1$. Let

$$f(x_1,\ldots,x_n) = p_1(x_1,\ldots,x_n)+\ldots+p_k(x_1,\ldots,x_n)$$

be a normal form representation of f as a sum of products. Now suppose that there is an l $(1\leqslant l\leqslant k)$ such that x_1 does not appear in p_l; as we have

$$a_1 = p_1(a_1,\ldots,a_n)+\ldots+p_l(a_2,\ldots,a_n)+\ldots+p_k(a_1,\ldots,a_n),$$

we get

$$a_1\geqslant p_l(a_2,\ldots,a_n)\geqslant \prod_{1<i\leqslant n} a_i.$$

Finally, we have to consider the case where x_1 appears in any of the p_j, and it never appears alone, since, otherwise, f would be trivial. Then we can pull out x_1, and we get

$$a_1\cdot\big(q_1(a_2,\ldots,a_n)+\ldots+q_k(a_2,\ldots,a_n)\big) = a_1$$

and, consequently,

$$a_1\leqslant \sum_{1<i\leqslant n} a_i.$$

This completes the proof of Theorem 1.

We can now give an easy example showing that M-independence does not coincide with T-independence in distributive lattices.

Example. In the power set of $\{1,2,\ldots,8\}$, consider the elements

$$X_1 = \{2,3,4,5\},\quad X_2 = \{1,3,4,6\},\quad X_3 = \{1,2,4,7\},$$
$$X_4 = \{1,2,3,8\}.$$

The set $\{X_1, X_2, X_3, X_4\}$ is not M-independent, since

$$X_1\cap X_2 = \{3,4\}\subseteq\{1,2,3,4,7,8\} = X_3\cup X_4.$$

But it is T-independent, as can easily be seen by help of Theorem 1.

Remark. It is obvious that T-independence does not coincide with C-independence in distributive lattices.

Definition 1. Let $(X;\ +,\cdot)$ be a distributive lattice, and $k\geqslant 1$ a positive integer. $I\subseteq X$ is called *T_k-independent* if for any different elements $a_1, \ldots, a_k, \ldots, a_n \in I$ $(n > k)$ we have

$$\prod_{1\leqslant i\leqslant k} a_i \nleqslant \sum_{k+1\leqslant j\leqslant n} a_j.$$

Let $K \subseteq N$; $I \subseteq X$ is called *T_K-independent* if it is T_k-independent for every $k \in K$.

So a finite set $\{a_1, \ldots, a_n\} \subseteq X$ is T-independent iff it is $T_{\{1,n-1\}}$--independent.

Definition 2. Let $K \subseteq N$ be an arbitrary non-void set, let $a = \sup K$, and $N_a := \{n \in N \mid n \leqslant a\}$. For $i \in N_a$ and $k \in K$, write

$$A_{ik} := \{A \subseteq N_a \mid |A| = k,\ i \in A\} \quad \text{and} \quad A_i := \bigcup_{k\in K} A_{ik}.$$

Then a distributive lattice P_K is defined by

$$P_K := \langle A_i \mid i \in N_a\rangle \leqslant 2^{2^{N_a}}.$$

LEMMA 1. *The set $\{A_i \mid i \in N_a\} \subseteq P_K$ is T_K-independent.*

If K is finite, it is not T_l-independent for any $l \in N_a - K$: if $l \in N_a - K$, then for any $A_{i_1}, \ldots, A_{i_l}$ there exist $A_{j_1}, \ldots, A_{j_m}$ $(i_r \neq j_s$ for any $r, s)$ such that

$$A_{i_1} \cap \ldots \cap A_{i_l} \subseteq A_{j_1} \cup \ldots \cup A_{j_m}.$$

Proof. Consider, for $k \in K$, any system $A_{i_1}, \ldots, A_{i_k}$; the set $\{i_1, \ldots, i_k\}$ is an element of $A_{i_1} \cap \ldots \cap A_{i_k}$, but it is not an element of any of the other A_i's. This proves the first part of the lemma.

Now let us consider $A_{i_1}, \ldots, A_{i_l}$ with $l \in N_a - K$, where K is finite. $A_{i_1} \cap \ldots \cap A_{i_l}$ is not empty, since there is a $k \in K$ with $k > l$. For each $N \in A_{i_1} \cap \ldots \cap A_{i_l}$, we take an $i_N \in N - \{i_1, \ldots, i_l\}$, so $N \in A_{i_N}$. By this, we get finitely many such sets A_{i_N} the join of which contains $A_{i_1} \cap \ldots \cap A_{i_l}$.

Remark. From Lemma 1 it follows that $P_{\{1,\ldots,n\}}$ is just the free distributive lattice on n generators, since an M-independent set $I \subseteq X$ generates a free algebra in the equational class generated by $(X; \mathscr{F})$.

THEOREM 2. *Let $K \subseteq N$ be finite and non-void, and let D be a finite distributive lattice generated by a T_K-independent set. Then there exists a homomorphism from D onto P_K.*

The proof of Theorem 2 will be a consequence of Lemmas 2 and 3, where it will be shown that the poset $J(P_K)$ of all join-irreducible elements of P_K can be isomorphically embedded into the poset $J(D)$; it is well known that such an embedding induces a homomorphism from D onto P_K.

COROLLARY 1. *Let D be a distributive lattice generated by a T-independent set $\{a_1, \ldots, a_n\}$. Then there exists a homomorphism from D onto $P_{\{1,n-1\}}$.*

LEMMA 2. *Let $K \subseteq N$ be non-void, and let D be a distributive lattice generated by a set I which is T_K-independent. Then, for any $k \in K$ and $a_1, \ldots, a_k \in I$, $a_1 \cdot \ldots \cdot a_k \in J(D)$.*

Proof. Let $p < a_1\cdot\ldots\cdot a_k$; as I generates D, we have $p = p_1\cdot\ldots\cdot p_r$, where each of the p_i is a sum of finitely many elements from I. There must be at least one of the p_i in which none of the a_j $(1 \leqslant j \leqslant k)$ occurs as a summand — otherwise, we would have $a_1\cdot\ldots\cdot a_k \leqslant p_i$ for each $1 \leqslant i \leqslant r$ and, consequently, $a_1\cdot\ldots\cdot a_k \leqslant p$. So let us assume that none of the a_j appears in p_1, say

$$p_1 = \sum_{1\leqslant t\leqslant l} a_t \quad \text{with } a_t \in I \text{ and } a_t \neq a_j \text{ for all } 1 \leqslant t \leqslant l \text{ and } 1 \leqslant j \leqslant k.$$

It follows then that

$$p \leqslant \sum_{1\leqslant t\leqslant l} a_t.$$

This and the T_k-independence of I imply that $a_1\cdot\ldots\cdot a_k$ cannot be the join of two smaller elements.

COROLLARY 2. *Let D be a distributive lattice generated by $\{a_1, \ldots, a_n\}$. Then $\{a_1, \ldots, a_n\}$ is T-independent iff each a_i is doubly irreducible and $\{a_1, \ldots, a_n\}$ is an antichain.*

Proof. From T_1-independence we can conclude the join-irreducibility of the a_i. As a T-independent set is also T-independent in the dual lattice, we know that each of the a_i is also meet-irreducible. The incomparability is immediate. To prove the other direction — as a_i is join-irreducible — we can conclude from $a_i \leqslant \sum_{j\neq i} a_j$ that $a_i \leqslant a_j$ for some $j \neq i$, contradicting the incomparability.

LEMMA 3. *Let $K \subseteq N$ be finite and non-void; then*

$$J(P_K) = \{A_{i_1}\cap\ldots\cap A_{i_k} \mid k \in K,\ i_j \leqslant a\}.$$

Proof. In view of Lemma 2 and distributivity, it is enough to show that, for $l \in N_a - K$ and arbitrary $A_{j_1}, \ldots, A_{j_l}$, $A_{j_1}\cap\ldots\cap A_{j_l}$ is a sum of elements of the form $A_{i_1}\cap\ldots\cap A_{i_k}$ with $k \in K$. But this follows from

$$A_{j_1}\cap\ldots\cap A_{j_l} = \{A \subseteq N_a \mid \{j_1, \ldots, j_l\} \subseteq A,\ |A| \in K\} = \bigcup_{\substack{\{j_1,\ldots,j_l\}\subseteq N\subseteq N_a \\ |N|\in K}} \bigcap_{n\in N} A_n.$$

Thus Lemma 3 is proved, and so is Theorem 2.

Remark. Theorem 2 says that, given a $K \subseteq N$, there is a "smallest" distributive lattice containing a T_K-independent set. Also, for a given $n \in N$, there is a "smallest" distributive lattice containing an n-element T-independent set (cf. Corollary 1). Finally, we give a simpler description of this lattice which we called $P_{\{1,n-1\}}$:

For $N_{2n} = \{1, 2, \ldots, 2n\}$, the sublattice of $2^{N_{2n}}$ generated by the elements $X_i = \{1, 2, \ldots, i-1, i+1, \ldots, n, n+i\}$, $1 \leqslant i \leqslant n$, is isomorphic to $P_{\{1,n-1\}}$. So, the lattice in the Example is isomorphic to $P_{\{1,3\}}$.

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