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SOME INTEGRAL INEQUALITIES OF STURM-LIOUVILLE TYPE

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1. Let us denote by *abs C* the class of real functions of a real variable t which are defined and absolutely continuous on the open interval $I = (a, \beta)$, $a < \beta$, bounded or not (¹). Let us put $v \equiv p\dot{\varphi}\varphi^{-1}$ and $q \equiv -(p\dot{\varphi})^{\cdot}\varphi^{-1}$, where functions p and φ belong to the class *abs C* and satisfy the conditions: $p > 0$, $\varphi > 0$ and $\dot{\varphi} \equiv d\varphi/dt \in abs\,C$. By $\hat{H}$ we mean the class of functions $h \in abs\,C$ satisfying the integral conditions (²)

$$\int_I qh^2\,dt > -\infty, \qquad \int_I p\dot{h}^2\,dt < \infty \tag{1.1}$$

and the limit conditions

$$\liminf_{t\to a} vh^2 < \infty, \qquad \limsup_{t\to\beta} vh^2 > -\infty. \tag{1.2}$$

THEOREM 1. *For every function $h \in \hat{H}$ both limits in* (1.2) *are proper and finite. Moreover, the equality*

$$\int_I p\dot{h}^2\,dt = \int_I qh^2\,dt + \int_I p\varphi^2\dot{f}^2\,dt + \lim_{t\to\beta} vh^2 - \lim_{t\to a} vh^2 \tag{1.3}$$

holds, where $f \equiv h\varphi^{-1}$.

Proof. It is easy to check that the identity

$$p\dot{h}^2 = qh^2 + p\varphi^2\dot{f}^2 + (vh^2)^{\cdot} \tag{$*$}$$

is valid a. e. in the interval I. It follows from the assumptions that functions $p\dot{h}^2$ and qh^2 are summable in every interval $\langle a, b\rangle$, where $a < a < b < \beta$. It follows further that $vh^2 \in abs\,C$. Thus the function $(vh^2)^{\cdot}$ is summable

(¹) A function $g\colon I \to R$ is *absolutely continuous* on an open interval I if it is absolutely continuous on every closed interval $\langle a, b\rangle \subset I$.

(²) A measurable function $g\colon I \to R$ is *integrable* in an interval I if at least one of the integrals $\int_I g^+\,dt$ and $\int_I g^-\,dt$ is finite, where $g^+ = \max(g, 0)$ and $g^- = \max(-g, 0)$. If both integrals are finite, then we say that the function g is *summable* in the interval I.

in the interval $\langle a, b\rangle$. Hence, by $(*)$, also the function $p\varphi^2\dot{f}^2$ is summable in $\langle a, b\rangle$ and we have

$$(**)\qquad \int_a^b p\dot{h}^2\,dt = \int_a^b qh^2\,dt + \int_a^b p\varphi^2\dot{f}^2\,dt + vh^2|_a^b.$$

Now, by conditions (1.2), there exist two sequences $\{a_n\}$ and $\{b_n\}$ such that $a < a_n < b_n < \beta$, $a_n \to a$, $b_n \to \beta$, and

$$\lim_{n\to\infty} vh^2|_{a_n} = A < \infty, \quad \lim_{n\to\infty}(-vh^2)|_{b_n} = B < \infty.$$

A priori the cases $A = -\infty$ and $B = -\infty$ are not excluded. In any case the sequence $-vh^2|_{a_n}^{b_n}$ is bounded from above by some finite constant C. Hence by $(**)$ we have the bound

$$\int_{a_n}^{b_n} qh^2\,dt \leqslant \int_a^\beta p\dot{h}^2\,dt + C,$$

because $p > 0$. From conditions (1.1) we conclude now that the function qh^2 is summable in the entire interval I.

In a similar way we prove that the function $p\varphi^2\dot{f}^2$ is summable in I. Hence all integrals in $(**)$ have finite limits as $a \to a$ or $b \to \beta$. This concludes the proof of Theorem 1 (cf. [12]).

With appropriate additional assumptions equality (1.3) represents the Weierstrass identity for a variational problem of Sturm and Liouville for a "field" of functions $h = f\varphi$, where the function φ satisfies the equation $(p\dot{\varphi})^{\cdot} + q\varphi = 0$ and some appropriate boundary conditions (see [5] and [8]). Observing the equality

$$\int_I p\varphi^2\dot{f}^2\,dt = \int_I (p\dot{h} - vh)^2 p^{-1}\,dt,$$

we notice that (1.3) may be treated as a modification of Beesack's identity (see [1]). We emphasize that the function φ need not belong to the class $\hat{H}$.

Remark 1. *The function φ belongs to the class $\hat{H}$ if and only if the following two conditions are satisfied:*

$$\text{(i)}\qquad \int_I p\dot{\varphi}^2\,dt < \infty,$$

$$\text{(ii)}\qquad \int_I |q|\varphi^2\,dt < \infty.$$

Condition (ii) *may be substituted by the conjunction of another two conditions:*

$$\text{(iii)}\qquad \int_I q^-\varphi^2\,dt < \infty,$$

(iv) *there exist finite limits of the expression $p\dot{\varphi}\varphi$ as $t \to a$ and $t \to \beta$.*

In fact, let us assume that $\varphi \in \hat{H}$. Hence condition (i) is satisfied. Condition (ii) is obtained directly from Theorem 1, where we have proved that the function qh^2 is summable in the interval I provided $h \in \hat{H}$. Further, condition (iii) follows from condition (ii). Using the identity

$$p\dot{\varphi}\varphi|_a^b = \int_a^b p\dot{\varphi}^2 dt - \int_a^b q\varphi^2 dt,$$

where $\alpha < a < b < \beta$, we prove that conditions (i) and (ii) imply condition (iv). Finally, by the definition of the class $\hat{H}$, the conjunction of conditions (i), (iii) and (iv) implies $\varphi \in \hat{H}$.

Let us notice that in the often appearing case of $q \geqslant 0$ condition (iii) is trivially satisfied.

THEOREM 2. *For every function $h \in \hat{H}$ the following inequality is valid*:

$$\lim_{t\to\beta} vh^2 - \lim_{t\to\alpha} vh^2 \leqslant \int_I (p\dot{h}^2 - qh^2)\,dt. \tag{1.4}$$

If $\varphi \notin \hat{H}$ and $h \not\equiv 0$, then (1.4) *is a strict inequality. If $\varphi \in \hat{H}$, then* (1.4) *becomes equality only in the case of $h = \mathrm{const}\cdot\varphi$.*

Proof. Inequality (1.4) follows from (1.3), because $p > 0$. If it becomes equality for some non-vanishing function $h \in \hat{H}$, then from (1.3) we have $(h\varphi^{-1})^{\cdot} = 0$ (a. e.), because $p\varphi^2 > 0$. But $h\varphi^{-1} \in absC$, hence $h = c\varphi$, where $c = \mathrm{const} \neq 0$. It implies that $\varphi \in \hat{H}$. Now it is easy to complete the proof (cf. [4] and [12]).

As an example of an application of Theorems 1 and 2 let us assume $I = (0, \infty)$, $p = 1$ and $\varphi = \exp(-\lambda t)$, where λ is an arbitrary positive constant. Then $v = -\lambda$ and $q = -\lambda^2$. We shall prove that the class $\hat{H}$ consists here of those functions $h \in absC$ which satisfy the conditions

$$\int_0^\infty h^2 dt < \infty, \quad \int_0^\infty \dot{h}^2 dt < \infty. \tag{1.5}$$

The necessity of (1.5) follows directly from (1.1). To prove that (1.5) is sufficient it remains to show that (1.5) implies (1.2). Since, in our case, we have $vh^2 = -\lambda h^2 \leqslant 0$, the first condition of (1.2) is surely satisfied. If the second condition of (1.2) were not satisfied, we would have $\lim_{t\to\beta} vh^2 = -\infty$, i. e. $\lim_{t\to\infty} h^2 = \infty$; a contradiction with the first condition of (1.5). Hence in the considered case conditions (1.5) represent a characterization of the class $\hat{H}$.

Let a function $h \in absC$ satisfy (1.5). From Theorem 1 it follows that the limits of the function h^2 as $t \to 0$ and $t \to \infty$ exist and are finite. By the first condition of (1.5) we must have $\lim_{t\to\infty} h^2 = 0$. We also get

$$\lim_{t\to 0} h^2 = (\lim_{t\to 0} h)^2,$$

because the function h is continuous for $t > 0$.

Using Theorem 2 and Remark 1 we infer that

If a function $h \in \operatorname{abs} C$ *satisfies* (1.5), *then there exists a limit value* $h(0) = \lim_{t\to 0} h$ *and the inequality*

$$h^2(0) \leqslant \lambda \int_0^\infty h^2 dt + \lambda^{-1} \int_0^\infty \dot{h}^2 dt \tag{1.6}$$

holds for every real number λ, $0 < \lambda < \infty$. *Inequality* (1.6) *becomes equality if and only if* $h = \mathrm{const}\cdot\exp(-\lambda t)$.

Finally, assume that $h \not\equiv 0$ and take the minimal value of the right-hand side of inequality (1.6) with respect to λ. To this aim we put $\lambda = \lambda_h > 0$, where

$$\lambda_h^2 = \Big(\int_0^\infty \dot{h}^2 dt\Big)\Big(\int_0^\infty h^2 dt\Big)^{-1}.$$

In this way we obtain an "optimal" bound

$$h^2(0) \leqslant 2\Big(\int_0^\infty h^2 dt\Big)^{1/2}\Big(\int_0^\infty \dot{h}^2 dt\Big)^{1/2} \tag{1.7}$$

(see [8], Theorem 263, and [4]). It becomes equality if and only if we put $h = \mathrm{const}\cdot\exp(-\lambda_h t)$, which does not impose any additional conditions with respect to λ_h besides $0 < \lambda_h < \infty$. In fact, after using the expression for h, the equation for λ_h becomes an identity.

We apply in the sequel the above technique of optimization for some inequalities containing a parameter.

2. Let H be the class of functions $h \in \operatorname{abs} C$ satisfying integral conditions (1.1) and limit conditions

$$\liminf_{t\to a} vh^2 \leqslant 0, \qquad \limsup_{t\to \beta} vh^2 \geqslant 0. \tag{2.1}$$

Obviously, $H \subset \hat{H}$. By Theorem 1, conditions (2.1) may be written equivalently in the form

$$\lim_{t\to a} vh^2 \leqslant 0, \qquad \lim_{t\to \beta} vh^2 \geqslant 0. \tag{2.1'}$$

THEOREM 3. *For every function* $h \in H$ *the inequality*

$$\int_I qh^2 dt \leqslant \int_I p\dot{h}^2 dt \tag{2.2}$$

holds. If $h \not\equiv 0$, *then inequality* (2.2) *becomes equality if and only if* $h\varphi^{-1} = \text{const} \neq 0$, *where the additional conditions*

$$\varphi \in \hat{H}, \quad \lim_{t\to a} p\dot{\varphi}\varphi = \lim_{t\to\beta} p\dot{\varphi}\varphi = 0 \tag{2.3}$$

must be satisfied (see Remark 1).

Proof. By virtue of limit conditions (2.1′) inequality (2.2) follows from inequality (1.4). If both sides of inequality (2.2) are equal for some non-vanishing function $h \in H$, then by (1.4) and (2.1′) we have

$$\lim_{t\to a} vh^2 = \lim_{t\to\beta} vh^2 = 0.$$

Applying Theorem 2 once again, we get $\varphi \in \hat{H}$ and $h = c\varphi$, where $c = \text{const} \neq 0$. This proves the validity of (2.3), because $v\varphi^2 = p\dot{\varphi}\varphi$. The theorem now follows easily.

We shall call inequalities of the form (2.2), which do not contain explicitly the limit conditions, the *inequalities of Sturm-Liouville type.* Let us notice that the condition $\varphi \in H$ is not sufficient for inequality (2.2) to become equality. If $q \geqslant 0$, then conditions (2.3) can be replaced by the following system of conditions (see Remark 1):

$$\int_I p\dot{\varphi}^2 dt < \infty, \quad \lim_{t\to a} p\dot{\varphi}\varphi = \lim_{t\to\beta} p\dot{\varphi}\varphi = 0. \tag{2.4}$$

As an example of an application of Theorem 3 take $I = (0, \infty)$, $p = 1$ and $\varphi = \exp(-\lambda t^2/2)$, where λ is an arbitrary positive constant. In such a case $v = -\lambda t$ and $q = \lambda - \lambda^2 t^2$. We shall prove that the class H consists here of those functions $h \in absC$ which satisfy the conditions

$$\int_0^\infty t^2 h^2 dt < \infty, \quad \int_0^\infty \dot{h}^2 dt < \infty. \tag{2.5}$$

Necessity. Let $h \in H$. As shown in the proof of Theorem 1, the function qh^2 is summable in the interval I. Thus we have

$$\int_0^\infty |\lambda - \lambda^2 t^2| h^2 dt < \infty.$$

After excluding the interval $\Delta = \{t\colon 2/3\lambda \leqslant t^2 \leqslant 2/\lambda\}$ from $(0, \infty)$ we obtain

$$\int_{(0,\infty)-\Delta} t^2 h^2 dt < \infty,$$

because $|\lambda - \lambda^2 t^2| \geqslant \lambda^2 t^2/2$ for $t \in (0, \infty) - \Delta$. Since the function $t^2 h^2$ is bounded in the interval Δ,

$$\int_\Delta t^2 h^2 dt < \infty.$$

Thus the first condition of (2.5) has been checked and the second condition of (2.5) follows directly from that of (1.1).

Sufficiency. If a function $h \in absC$ satisfies conditions (2.5), then it satisfies the second condition of (1.1) and also the first condition of (2.1), because $vh^2 = -\lambda th^2 \leqslant 0$ for $t > 0$. We find further

$$\int_I q^- h^2 dt = \int_{t_0}^{\infty} (\lambda^2 t^2 - \lambda) h^2 dt \leqslant \lambda^2 \int_{t_0}^{\infty} t^2 h^2 dt \leqslant \lambda^2 \int_0^{\infty} t^2 h^2 dt,$$

where $t_0 = \lambda^{-1/2}$, which by (2.5) assures that the first condition of (1.1) is fulfilled. It remains to consider the second condition of (2.1). If a function h did not satisfy the second condition of (1.2), then we would have

$$\lim_{t\to\beta} vh^2 = -\infty, \quad \text{i. e.} \quad \lim_{t\to\infty} th^2 = \infty,$$

which is a contradiction with the first condition of (2.5). Surely, we have $h \in \hat{H}$. Hence, by Theorem 1, there exists a finite limit $\lim_{t\to\beta} vh^2$, i. e. $\lim_{t\to\infty} th^2$, and by the first condition of (2.5) it must be $\lim_{t\to\infty} th^2 = 0$. In other words, we get $\lim_{t\to\beta} vh^2 = 0$ and so also the second condition of (2.1) is valid. Thus we have shown that conditions (2.5) are a characterization of the class H. It is easy to see that $H = \hat{H}$ in our case.

Now, applying Theorem 3 we infer that

If a function $h \in absC$ satisfies conditions (2.5), *then the inequality*

$$\int_0^{\infty} h^2 dt \leqslant \lambda \int_0^{\infty} t^2 h^2 dt + \lambda^{-1} \int_0^{\infty} \dot{h}^2 dt \tag{2.6}$$

is valid for every real number λ, $0 < \lambda < \infty$. Inequality (2.6) *becomes equality if and only if* $h = \text{const}\cdot\exp(-\lambda t^2/2)$.

Applying the described optimization to (2.6) with respect to λ we obtain the bound

$$\Big(\int_0^{\infty} h^2 dt\Big)^2 \leqslant 4 \int_0^{\infty} t^2 h^2 dt \cdot \int_0^{\infty} \dot{h}^2 dt, \tag{2.7}$$

which becomes equality if and only if $h = \text{const}\cdot\exp(-\lambda t^2/2)$.

Clearly, (2.7) represents the well-known Weyl inequality (see [4], [8], Theorem 226, and [11]). With the additional condition $h(0) = 0$ (i. e., $\lim_{t\to 0} h = 0$) Weyl's inequality may be improved to the form

$$\Big(\int_0^{\infty} h^2 dt\Big)^2 \leqslant \frac{4}{9} \int_0^{\infty} t^2 h^2 dt \cdot \int_0^{\infty} \dot{h}^2 dt, \tag{2.8}$$

which becomes equality (finite) if and only if $h = \text{const}\cdot t\exp(-\lambda t^2/2)$, $0 < \lambda < \infty$. To get result (2.8) one may take the interval $I = (0, \infty)$,

$p = 1$ and $\varphi = t\exp(-\lambda t^2/2)$ and proceed as in the second example above. The only difference is that the validity of the first condition of (2.1′) requires $\lim_{t\to a} vh^2 = 0$ to be proved, because otherwise we would have $v = t^{-1} - \lambda t$, hence $v > 0$ in a right neighbourhood of the point 0. However, the equality $\lim_{t\to 0} t^{-1}h^2 = 0$ follows directly from $h(0) = 0$ and from the estimation

$$h^2 \leqslant t\int_0^t \dot{h}^2 dt.$$

The examples above show that the determination of the classes $\hat{H}$ and H for some non-trivial applications of Theorems 2 and 3 may be troublesome. However, in paper [3] various groups of assumptions assuring the relation $h \in H$ are described. In the sequel we discuss the problem more closely.

3. At first we prove

LEMMA 1. *Let a given function $h \in \operatorname{abs} C$ satisfy the condition $\int_I p\dot{h}^2 dt < \infty$. If the integral $\int_I p^{-1}dt$ is convergent at a point a* [3] *(resp. at a point β), then there exists a finite limit value $h(a) = \lim_{t\to a} h$ (resp. $h(\beta) = \lim_{t\to\beta} h$).*

Proof. Using the Schwarz inequality we obtain the estimation

$$[h(b)-h(a)]^2 = \Big(\int_a^b \dot{h}\,dt\Big)^2 \leqslant \int_a^b p^{-1}dt\cdot\int_a^b p\dot{h}^2 dt, \tag{3.1}$$

where $\alpha < a < b < \beta$. Lemma 1 follows now from the Cauchy condition for the existence of the limit.

As we have already said, the inequality $q \geqslant 0$ assures that the first condition of (1.1) is valid for an arbitrary measurable function $h\colon I\to R$. We shall assume that $q > 0$ (a. e.) and prove

LEMMA 2. (i) *The function v is decreasing in the interval I, and so the limit values $v(a) = \lim_{t\to a} v$ and $v(\beta) = \lim_{t\to\beta} v$ exist, finite or not; moreover, $v(a) > v(\beta)$.*

(ii) *If $v(a) \neq 0$ (resp. $v(\beta) \neq 0$), then the integral $\int_I p^{-1}dt$ is convergent at a point a (resp. at a point β) and*

$$v\int_a^t p^{-1}dt = O(1) \quad \textit{as } t\to a$$

(resp. $v\int_t^\beta p^{-1}dt = O(1)$ as $t\to\beta$).

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[3] The integral $\int_I g\,dt$ is *convergent* at a point $t_0 \in \bar{I}$ if the function g is summable in some neighbourhood of t_0.

Proof. First notice that v belongs to the class $abs\,C$ and it satisfies the Ricatti equation

$$\dot{v}+p^{-1}v^2+q = 0 \tag{3.2}$$

(cf. [1], [2] and [4]). Thus we have $-\dot{v} \geqslant q$. Integrating, we find

$$v(a)-v(b) \geqslant \int_a^b q\,dt > 0$$

for $\alpha \leqslant a < b \leqslant \beta$, because $q > 0$ (a. e.) which proves (i). To prove (ii), consider a neighbourhood of α in which $v \neq 0$. By (3.2) and $q \geqslant 0$ we have the estimation

$$\int_a^t p^{-1}dt \leqslant -\int_a^t v^{-2}\dot{v}\,dt = v^{-1}(t)-v^{-1}(a) \tag{3.3}$$

for $\alpha < a < t < \beta$ in that neighbourhood of α. Hence it follows immediately that

$$\int_\alpha^t p^{-1}dt < \infty,$$

because $v(\alpha) \neq 0$. If $v(\alpha) \neq \infty$, then the second part of (ii) is evident. And if $v(\alpha) = \infty$, then by (3.3) we have

$$\int_\alpha^t p^{-1}dt \leqslant v^{-1}(t)$$

and therefore

$$v\int_\alpha^t p^{-1}dt \leqslant 1,$$

which completes the proof.

We denote by H_0 (resp. H^0) the class of functions $h \in abs\,C$ satisfying the integral condition

$$\int_I p\dot{h}^2 dt < \infty$$

and the limit condition

$$\lim_{t\to\alpha} h = 0 \qquad (\text{resp. } \lim_{t\to\beta} h = 0).$$

LEMMA 3. (i) *If* $v(\beta) \geqslant 0$, *then* $H \subset H_0$.

(ii) *If* $v(\alpha) \leqslant 0$, *then* $H \subset H^0$.

(iii) *If* $v(\alpha) > 0$ *and* $v(\beta) < 0$, *then* $H \subset H_0 \cap H^0$.

Proof. We shall prove only (i). To this end we take $h \in H$ and $v(\beta) \geqslant 0$. Then $v(\alpha) > 0$ and the integral $\int_I p^{-1}dt$ must be convergent at a point α by

Lemma 2. Further, by Lemma 1, there exists a finite limit value $h(a) = \lim\limits_{t\to a} h$. Let us suppose that $h(a) \neq 0$. Hence

$$\lim_{t\to a} vh^2 = v(a)h^2(a) > 0,$$

which contradicts the first condition of (2.1′). Thus $h(a) = 0$ and $h \epsilon H_0$, which completes the proof.

We assume the following terminology:

a boundary point a (resp. β) of the interval I is *free* if $v(a) \leqslant 0$ (resp. $v(\beta) \geqslant 0$);

a boundary point a (resp. β) of the interval I is *fixed* if $v(a) > 0$ (resp. $v(\beta) < 0$).

THEOREM 4. *The following statements are valid under the assumption $q > 0$ (a. e.):*

(i) *If the point a is fixed and the point β is free, then $H = H_0$.*

(ii) *If the point a is free and the point β is fixed, then $H = H^0$.*

(iii) *If both points a, β are fixed, then $H = H_0 \cap H^0$.*

Proof. We prove only (i). By means of Lemma 3 it is sufficient to show that $H_0 \subset H$. By Lemma 2(i) there is $v \geqslant 0$, because $v(\beta) \geqslant 0$, and so the second condition of (2.1) is surely satisfied. Further, if $v(a) > 0$, then we use the estimation

$$0 \leqslant vh^2 \leqslant v \int_a^t p^{-1} dt \cdot \int_a^t p\dot{h}^2 dt$$

which follows from estimation (3.1) for $a \to a$ and $b = t > a$. As one can see we have $\lim\limits_{t\to a} vh^2 = 0$ only if

$$v \int_a^t p^{-1} dt = O(1) \quad \text{as } t \to a.$$

Hence, by Lemma 2(ii), the first condition of (2.1) is satisfied. Thus we may conclude that $h \epsilon H$.

Let us notice that under assumption $q > 0$ it follows from Lemma 2(i) that the two end-points of the interval I cannot be simultaneously free. If $v(a) > 0$ and $v(\beta) < 0$, then there exists a point $\gamma \epsilon I$ such that $v(\gamma) = 0$. By an appropriate partition of the interval I into two parts, part (iii) of Theorem 4 may be reduced to parts (i) and (ii).

4. In this section we consider some special cases of Theorem 3. We have throughout $q > 0$ and so the class H is determined by Theorem 4. The results are given in Table 1. To every row with a number n there corresponds, according to Theorem 3, some Sturm-Liouville type inequality which we shall call "the inequality n". In the column "$=$" we have marked by $+$ (or $-$) if conditions (2.4) are (or are not) satisfied. The last column contains references.

Inequalities 1, 2, 3 *and* 4. It is easy to see that inequalities 1 and 3 are equivalent to each other and equivalent to Wirtinger's inequality. Inequality 3 is sometimes called first Steklov's inequality. Inequalities 2 and 4 are generalizations of inequalities 1 and 3.

Inequalities 5, 6, 7 *and* 8. Inequalities 5 and 6 are two equivalent forms of the known Hardy's integral inequality. Inequalities 7 and 8 are its trivial generalizations.

Putting $I = (0, \infty)$, $p = 1$ and $\varphi = t^{1/2}\exp(-\lambda t)$ and using the same method as in the example of Section 2 we obtain the optimal (with respect to λ) bound of the form

$$4\int_0^\infty \dot{h}^2 dt - \int_0^\infty t^2 h^2 dt > \Big(\int_0^\infty t^{-1} h^2 dt\Big)^2 \Big(\int_0^\infty h^2 dt\Big)^{-1}, \tag{4.1}$$

which is valid for every non-vanishing function $h \in H_0$. Inequality (4.1) represents a certain modification of Hardy's inequality that states only the positivity of its left-hand side. Clearly, using equality (1.3) one can evaluate the value of this left-hand side but the derivative $\dot{h}$ that is not present on the right-hand side of (4.1) has to be used.

Another type of a modification of Hardy's inequality is obtained in the case of $I = (0, \beta)$, where $0 < \beta < \infty$, $p = 1$ and $\varphi = t^\lambda$, by means of Theorem 2 and the optimization technique with respect to λ (presented in Section 2). After evaluation we get the bound

$$\frac{1}{\beta} h^2(\beta) + \int_0^\beta t^{-2} h^2 dt \leqslant 2\Big(\int_0^\beta \dot{h}^2 dt\Big)^{1/2} \Big(\int_0^\beta t^{-2} h^2 dt\Big)^{1/2} \tag{4.2}$$

which is valid for every function $h \in H_0$, where $h(\beta) = \lim\limits_{t\to\beta} h$. The existence of that limit follows from Lemma 1 in Section 3. One can show that inequality (4.2) is a generalization of the inequality presented in [8], Theorem 254 (cf. [12]).

Inequality 9. Simultaneously with that inequality one can consider a "mirror" inequality which is obtained from inequality 9 by putting $-t$ instead of t. Using both inequalities we get the following corollary:

For an arbitrary function h that is absolutely continuous in the interval $(-1, +1)$ *and satisfies the condition*

$$\int_{-1}^{+1} (1-t^2)\dot{h}^2 dt < \infty$$

the following inequality is valid:

$$\int_{-1}^{+1} [h-h(0)]^2 dt \leqslant \frac{1}{2} \int_{-1}^{+1} (1-t^2)\dot{h}^2 dt. \tag{4.3}$$

It becomes equality if and only if $h = a+bt$ for $t \geqslant 0$ and $h = a+ct$ for $t \leqslant 0$, where a, b, c are arbitrary constants.

Evidently we further obtain

$$\int_{-1}^{+1} [h-h(0)]^2 dt \geqslant \int_{-1}^{+1} h^2 dt - \frac{1}{2}\left(\int_{-1}^{+1} h\,dt\right)^2, \tag{4.4}$$

because the right-hand side of this inequality is a minimal value of the integral $\int_{-1}^{+1} (h-\lambda)^2 dt$ with respect to λ. Inequality (4.4) becomes equality if and only if

$$h(0) = \frac{1}{2} \int_{-1}^{+1} h\,dt.$$

From inequalities (4.3) and (4.4) we have the inequality

$$\int_{-1}^{+1} h^2 dt - \frac{1}{2}\left(\int_{-1}^{+1} h\,dt\right)^2 \leqslant \frac{1}{2} \int_{-1}^{+1} (1-t^2)\dot{h}^2 dt. \tag{4.5}$$

It becomes equality if and only if both inequalities (4.3) and (4.4) are equalities which defines the form of the function h, namely it must be $h = a+bt$ for $-1 < t < +1$, where a and b are arbitrary constants. In this way inequality 9 yields a known inequality ([8], Theorem 225).

Similarly, from inequality 1 we have the inequality

$$\int_{-\pi/2}^{+\pi/2} h^2 dt - \frac{1}{\pi}\left(\int_{-\pi/2}^{+\pi/2} h\,dt\right)^2 \leqslant \int_{-\pi/2}^{+\pi/2} \dot{h}^2 dt \tag{4.6}$$

valid for every function h which is absolutely continuous in the interval $(-\pi/2, +\pi/2)$ and for which the integral on the right-hand side is finite. Inequality (4.6) becomes equality if and only if $h = a+b\sin t$, where a and b are arbitrary constants. Assuming moreover

$$\int_{-\pi/2}^{+\pi/2} h\,dt = 0$$

we get from (4.6) the second Steklov inequality.

TABLE 1

n	$I = (a, \beta)$	p	φ'	q	H	$=$	References
1	$\left(0, \frac{\pi}{2}\right)$	1	$\sin t$	1	H_0	+	[8, Th. 256] [1], [4]
2	$\left(0, \frac{\pi}{2}\right)$	$(\cos t)^a$ $a > -1$	$\sin t$	$(a+1)(\cos t)^a$	H_0	+	[4], [12]
3	$(0, \pi)$	1	$\sin t$	1	$H_0 \cap H^0$	+	[8, Th. 257], [4]
4	$(0, \pi)$	$\lvert\cos t\rvert^a$ $a > -1$	$\sin t$	$(a+1)\lvert\cos t\rvert^a$	$H_0 \cap H^0$	+	[12]
5	$(0, \infty)$	1	$t^{1/2}$	$\frac{1}{4t^2}$	H_0	−	[8, Th. 253], [2], [4]
6	$(0, \infty)$	t^2	$t^{-1/2}$	$\frac{1}{4}$	H^0	−	[8, Th. 328], [2]
7	$(0, \beta)$ $0 < \beta \leqslant \infty$	t^a $a \neq 1$	$t^{(1-a)/2}$	$\frac{(1-a)^2}{4} t^{a-2}$	H_0; $a < 1$ H^0; $a > 1$	−	[8, Th. 330], [3], [6]
8	$0 \leqslant a < \beta \leqslant \infty$	1	t^a $0 < a < 1$	$\frac{a(1-a)}{t^2}$	H_0	−	[12]
9	$(0, 1)$	$1-t^2$	t	2	H_0	+	[4]
10	$(-1, 1)$	1	$(1-t^2)^{1/2}$	$\frac{1}{(1-t^2)^2}$	$H_0 \cap H^0$	−	[1]
11	$(-1, 1)$	$(1-t^2)^a$ $a < 1/2$	$(1-t^2)^{1/2-a}$	$(1-2a)(1-t^2)^{a-2}$	$H_0 \cap H^0$	+ $a < 0$	[9], [12]
12	$(-1, 1)$	1	$1-t^2$	$\frac{2}{1-t^2}$	$H_0 \cap H^0$	+	[8, Th. 262], [1], [4]

13	$(-1, 1)$	$(1-t^2)^a$ $a < 1$	$(1-t^2)^{1-a}$	$(2-2a)(1-t^2)^{a-1}$	$H_0 \cap H^0$	$+$	[9], [12]
14	$(0, \infty)$	$(1+t^2)^a$ $a < \frac{3}{2}$	$t(1+t^2)^{-1/2}$	$(3-2a)(1+t^2)^{a-2}$	H_0	$+$	[1], [12]
15	$(0, \infty)$	$\frac{1}{a-b}(e^{at}-e^{bt})$ $a > 0, b \neq a$	e^{-at}	ae^{bt}	H^0	$+$ $b < 2a$	[12]
16	$-\infty < \alpha < \beta < \infty$	e^{-at} $a \neq 0$	$e^{at/2}$	$\frac{1}{4}a^2e^{-at}$	$H_0;\ a > 0$ $H^0;\ a < 0$	$-$	[4]
17	$0 < \alpha < \beta < \infty$	e^{-at^2} $a > 0$	t	$2ae^{-at^2}$	H_0	$+$ $\alpha = 0$ $\beta = \infty$	[4]

Inequalities 10, 11, 12 *and* 13. They represent a complete system of Sturm-Liouville type inequalities obtained by substitutions

$$p = (1-t^2)^a, \quad \varphi = (1-t^2)^\beta, \quad q = \lambda(1-t^2)^\gamma,$$

where a, β, γ and $\lambda > 0$ are constants. In certain problems of the non-linear oscillation theory, inequality 11 for $a = -\frac{1}{2}$ (see [13]) is of a great use. This was the reason for a closer study of this inequality with additional assumption of orthogonality

$$\int_{-1}^{+1} q\varphi h\, dt = 0$$

(see [7], [10]).

A similar complete system of inequalities may be obtained by more general substitutions, namely

$$p = (1-t)^a (1+t)^b, \quad \varphi = (1-t)^k (1+t)^l, \quad q = \lambda(1-t)^\alpha (1+t)^\beta,$$

where a, b, k, l, α, β and λ are constants. The constants should be chosen in such a way that the equation $(p\dot{\varphi})^{\cdot} + q\varphi = 0$ be satisfied. It is easy to show that with arbitrary values of the constants a and b the constants k, l and λ exist only for α and β given in Table 2.

TABLE 2

	I		II		III	
α	$a-1$	$a-2$	a	$a-2$	$a-1$	$a-2$
β	$b-1$	$b-2$	$b-2$	b	$b-2$	$b-1$

Constants a and b should satisfy the additional condition $\lambda > 0$.

We omit the evaluation, noting only that case I covers inequalities 10-13 with the assumption $a = b$ (see [12]).

Inequality 14. This inequality was deduced in [1] for $a = 0$ under additional assumption $h^2 = O(t^{3-\varepsilon})$ as $t \to \infty$. In the case of $h(0) = 0$, which is of interest to us, this assumption is trivially satisfied (cf. [12]). Evidently, convergence of the integral $\int_0^\infty \dot{h}^2 dt$ implies $h^2 = o(t)$ as $t \to \infty$ (see [8], Theorem 223).

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