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A CERTAIN GENERALIZATION OF A RESULT OF ERDÖS AND SURÁNYI

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P. Erdös and J. Surányi proved [1] that every positive integer n can be written in the form

$$n = \sum_{m=1}^{k} \varepsilon_m m^2$$

with suitable k and $\varepsilon_m = \pm 1$ $(1 \leqslant m \leqslant k)$. We shall show here that we can replace in this result the squares by any power.

THEOREM. *For any integer $n \geqslant 1$ and any $r \geqslant 2$ we can find an integer $k \geqslant 1$ and $\varepsilon_1, \ldots, \varepsilon_k \in \{\pm 1\}$ such that*

$$n = \sum_{m=1}^{k} \varepsilon_m m^r. \tag{1}$$

Proof. Consider the sequence $w_0(x), w_1(x), \ldots$ of polynomials defined by

$$\begin{aligned} w_0(x) &= (x+1)^r - x^r, \\ w_{i+1}(x) &= w_i(x+2^{i+1}) - w_i(x) \quad (i = 1, 2, \ldots). \end{aligned}$$

We see immediately that with suitable $c_d(r, i) = \pm 1$ we can write

$$w_i(x) = \sum_{d=1}^{2^{i+1}} c_d(r, i)(x+d-1)^r \quad (i = 0, 1, \ldots).$$

We also have $\deg w_i = r-i-1$ $(i = 0, 1, \ldots, r-1)$, $w_{r-1}(x) = N_r = r!2^{r(r-1)/2}$, and $w_j(x) = 0$ for $j \geqslant r$.

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(1) W. Sierpiński, *250 problems n elementary number theory,i* Problem 250, New York - Warszawa 1970.

LEMMA. *If c is a positive integer and for a sequence $\eta_1, \ldots, \eta_s \in \{\pm 1\}$ we have*

$$\sum_{m=1}^{s} \eta_m m^r \equiv c \pmod{N_r},$$

then the integer c is representable in the form $\sum_{j=1}^{t} \varepsilon_j j^r$ for a certain $t \geqslant 1$ and $\varepsilon_1, \ldots, \varepsilon_t \in \{\pm 1\}$.

Proof. The assumptions imply that for a certain integer k we have

$$\sum_{m=1}^{s} \eta_m m^r = c + kN_r,$$

whence

$$c = \sum_{m=1}^{s} \eta_m m^r - kN_r = \sum_{m=1}^{s} \eta_m m^r - (\operatorname{sign} k) \sum_{j=0}^{|k|-1} w_{r-1}(s+j\cdot 2^r+1)$$

$$= \sum_{m=1}^{s} \eta_m m^r - \sum_{j=0}^{|k|-1} \sum_{d_j=1}^{2^r} (\operatorname{sign} k) c_{d_j}(r, r-1)(s+j\cdot 2^r+d_j)^r,$$

and this gives the required representation of c.

Thus it suffices to show that there is a number congruent to c $(\operatorname{mod} N_r)$ which is representable in the form (1).

For a given u consider now the number

$$V = \sum_{l=0}^{u-1} v(2lN_r),$$

where

$$v(x) = \sum_{k=1}^{N_r} \{(x+N_r+k)^r - (x+k)^r\} + 2(x+1)^r.$$

Evidently,

$$V \equiv \sum_{l=0}^{u-1} v(0) \equiv uv(0) \equiv 2u \pmod{N_r}.$$

On the other hand, our construction shows that

$$V = \sum_{j=1}^{2uN_r} \varepsilon_j j^r \tag{2}$$

holds with suitable $\varepsilon_1, \ldots, \varepsilon_{2uN_r} \in \{\pm 1\}$.

If c is even, then taking $u = c/2$ we obtain our assertion. If however c is odd, then we take $u = (c-1)/2$ to obtain (2), and then note that

$$c \equiv 2u+1 \equiv \sum_{j=1}^{2uN_r} \varepsilon_j j^r + (1+2uN_r)^r \pmod{N_r}.$$

This completes the proof.

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