

*DIVISIBILITY PROPERTIES
OF A CLASS OF MULTIPLICATIVE FUNCTIONS*

BY

W. NARKIEWICZ (WROCŁAW)

1. Let $f(n)$ be an integer-valued multiplicative function having the following property: there exists a polynomial $W(x)$ with integral coefficients such that, for all primes p , $f(p) = W(p)$. (Nearly all functions usually used in number theory have this property.) For any prime q , let $\beta(q)$ be the number of integers n ($1 \leq n \leq q-1$) such that $W(n) \not\equiv 0 \pmod{q}$ and let R_f be the set of all primes q with $\beta(q) = q-1$. If q^m divides N but q^{m+1} does not, then we write $q^m \parallel N$. In particular, $q^0 \parallel N$ means the same as $q \nmid N$. Finally, by $N(n \leq x: P)$ we denote the number of positive integers $n \leq x$ with the property P .

Generalizing a result of Scourfield [10], who considered the functions $f(n) = d(n)$, $\varphi(n)$ and $\sigma_r(n)$, we prove the following

THEOREM I. *Let q be a prime and m a non-negative integer such that the set $\{n: q^m \parallel f(n)\}$ is not void. Then the following asymptotic evaluations hold:*

(i) *if $q \notin R_f$, $\beta(q) \neq 0$, then*

$$N(n \leq x: q^m \parallel f(n)) \sim C_1(q, m)x(\log \log x)^M(\log x)^a,$$

where $a = \beta(q)/(q-1) - 1$;

(ii) *if $q \notin R_f$, $\beta(q) = 0$, $M(q, m) \neq 0$, then*

$$N(n \leq x: q^m \parallel f(n)) \sim C_2(q, m)x(\log \log x)^{M-1}(\log x)^{-1};$$

(iii) *if $q \notin R_f$, $\beta(q) = M(q, m) = 0$, then*

$$N(n \leq x: q^m \parallel f(n)) = O(x^{1/2});$$

(iv) *if $q \in R_f$, then*

$$N(n \leq x: q^m \parallel f(n)) \sim C_3(q, m)x.$$

Here $M = M(q, m) \leq m$ is defined in Section 3 and $C_i(q, m)$ are positive constants ($i = 1, 2, 3$). The value of $C_3(q, 0)$ is given by formula (3) in Section 3.

It should be observed that the set $\{n: q^0 \parallel f(n)\}$ is not void, as $q \nmid 1 = f(1)$ by the multiplicativity of $f(n)$. In the case (iii) of Theorem I one can obtain asymptotic equality under more stringent assumption about $f(n)$, but we will not go further into this subject.

Theorem I implies the following

COROLLARY. *If D is an integer without a prime divisor in the set R_f , then, for almost all n , D divides $f(n)$.*

Indeed, if $D = p_1^{a_1} \dots p_k^{a_k}$ with $p_i \notin R_f$, then

$$N(n \leq x: D \nmid f(n)) \leq \sum_{j=1}^k N(n \leq x: p_j^{a_j} \nmid f(n)),$$

but

$$N(n \leq x: p_j^{a_j} \nmid f(n)) = \sum_{i=0}^{a_j-1} N(n \leq x: p_j^i \parallel f(n)) = o(x)$$

by Theorem I and the corollary follows.

If D is square-free, it is possible to get more information about $N_D(x) = N(n \leq x: D \nmid f(n))$. In fact, we shall prove

THEOREM II. (i) *If D is a square-free integer which has no prime divisor in R_f , then either*

$$(a) \quad N_D(x) \sim C(D)x(\log x)^{-A}$$

with positive A and $C(D)$, or

$$(b) \quad N_D(x) = O(x^{1-\varepsilon}) \text{ with a suitable } \varepsilon > 0.$$

(ii) *If D is a square-free integer having prime divisors in R_f , then*

$$(c) \quad N_D(x) \sim C(D)x$$

with $C(D)$ positive.

In the cases of $f(n) = \sigma_r(n)$, $\varphi(n)$ or $d(n)$, Theorem I and some special cases of Theorem II were proved by Scourfield [10]. (Note that Theorem II implies that in corollary 3 on p. 282 of [10] one of the inequality signs can be replaced by equality sign provided that D is square-free.) For the function $\sigma_r(n)$ with odd r , Watson [11] proved earlier the corollary stated above, which enabled him to prove a statement of Ramanujan, and Rankin [6] proved Theorem I for $m = 0$ and also some cases of Theorem II. The function $f(n) = d(n)$ was considered by Sathe [7], [8], [9], who, among other results, proved that the constant $C_3(q, 0)$ in Theorem I, (iv) is in this case equal to $\zeta(q)/\zeta(q-1)$, if q is an odd prime (cf. also [1] and [2], where Cohen gave elementary proofs for $f(n) = d(n)$).

Our proof is based on two lemmas of Scourfield and a tauberian theorem due to Delange [3] (cf. also theorem 3 in [10]). It should be noted that the case $m = 0$ of Theorem I can be dealt with in an elementary way, using a theorem of Wirsing [12].

In Section 2 we collect the needed tools, in Section 3 we prove Theorem I and Section 4 contains the proof of Theorem II and some examples.

I am indebted to A. Schinzel for valuable remarks concerning this paper.

2. We shall use a generalization of Ikehara's theorem due to Delange [3] which we state as

LEMMA 1. (i) If $a_n \geq 0$, b is a real number and, for $\operatorname{Re} s > b$, we have

$$\sum_{n=1}^{\infty} a_n n^{-s} = (s-b)^{-a} \sum_{j=0}^m g_j(s) \log^j(1/(s-b)),$$

where a is positive, $g_0(s), \dots, g_m(s)$ are regular for $\operatorname{Re} s \geq b$ and $g_m(b) \neq 0$, then

$$\sum_{n \leq x} a_n \sim b^{-1} \cdot g_m(b) \cdot \Gamma(a)^{-1} x^b (\log \log x)^m (\log x)^{a-1}.$$

(ii) If $a_n \geq 0$ and

$$\sum_{n=1}^{\infty} a_n n^{-s} = g_0(s)(s-1)^{-a} + \sum_{j=1}^m g_j(s)(s-1)^{-c_j} + h(s),$$

for $\operatorname{Re} s > 1$, where a is a real number not equal to zero or a negative integer, $g_0(s), \dots, g_m(s), h(s)$ are regular for $\operatorname{Re} s \geq 1$, $g_0(1) \neq 0$, and the c_j 's are real numbers not equal to zero or a negative integer, all less than a , then

$$\sum_{n \leq x} a_n \sim g_0(1) \cdot \Gamma(a)^{-1} x (\log x)^{a-1}.$$

(iii) If $a_n \geq 0$ and

$$\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{j=0}^m g_j(s) \log^j(1/(s-1))$$

for $\operatorname{Re} s > 1$, where $g_0(s), \dots, g_m(s)$ are regular for $\operatorname{Re} s \geq 1$, $m \geq 1$ and $g_m(1) \neq 0$, then

$$\sum_{n \leq x} a_n \sim m g_m(1) x (\log \log x)^{m-1} (\log x)^{-1}.$$

Moreover, we shall use some results proved by Scourfield in [10] for the case $f(n) = \sigma_\nu(n)$, the proof of which uses only the multiplicativity of $\sigma_\nu(n)$ and so carries through without change to our more general

situation. To state them we have to introduce some notation which will be preserved in the sequel. Let q be a rational prime and let

$$a_m(n) = \begin{cases} 1 & \text{if } q^m \parallel f(n), \\ 0 & \text{if } q^m \nmid f(n) \end{cases}$$

for $m = 0, 1, 2, \dots$. Moreover, let

$$\Phi_m(p, s) = \sum_{j=1}^{\infty} a_m(p^j) p^{-js}$$

for $\operatorname{Re} s > 1$ and p prime.

Following [10] we denote by $\mathcal{R}(m)$ the set of all systems (r_1, r_2, \dots, r_k) with $r_i \geq 1$ and $r_1 + \dots + r_k = m$. (We do not regard as different two systems which differ only in ordering.) If $R \in \mathcal{R}(m)$ and R has t distinct elements occurring m_1, \dots, m_t times, respectively, then we have $F(R) = (m_1! \dots m_t!)^{-1}$.

LEMMA 2 (Scourfield [10], Lemma 8 and 9). *For $m \geq 1$ and $\operatorname{Re} s > 1$ we have*

$$\begin{aligned} & \sum_{n=1}^{\infty} a_m(n) n^{-s} \\ &= \sum_{n=1}^{\infty} a_0(n) n^{-s} \cdot \sum_{R=(r_1, \dots, r_k) \in \mathcal{R}(m)} F(R) \sum_{\langle p_1, \dots, p_k \rangle}^* \prod_{i=1}^k \Phi_{r_i}(p_i; s) (1 + \Phi_0(p_i, s))^{-1}, \end{aligned}$$

where the asterisk indicates that the summation is extended over all ordered systems (p_1, \dots, p_k) of distinct primes.

In the sequel, the letter p with or without indices is reserved for primes exclusively.

3. First we prove Theorem I in the case of $m = 0$. The idea of our proof is due to Rankin who proved this theorem for $f(n) = \sigma_r(n)$ in [6].

As $a_0(n)$ is multiplicative, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} a_0(n) n^{-s} \\ &= \prod_p (1 + \Phi_0(p, s)) = \prod_p (1 + \Phi_0(p, s)) \exp\{-a_0(p)p^{-s}\} \cdot \prod_p \exp\{a_0(p)p^{-s}\} \end{aligned}$$

for $\operatorname{Re} s > 1$.

The factor $\prod_p (1 + \Phi_0(p, s)) \exp\{-a_0(p)p^{-s}\}$ represents a function regular for $\operatorname{Re} s > 1/2$, and

$$\begin{aligned} \prod_p \exp\{a_0(p)p^{-s}\} &= \exp\left\{\sum_p a_0(p)p^{-s}\right\} \\ &= \exp\left\{a_0(q)q^{-s} + \sum_{j=1}^{q-1} \varepsilon(j) \sum_{p \equiv j \pmod{q}} p^{-s}\right\}, \end{aligned}$$

where $\varepsilon(j) = 1$ if $W(j) \not\equiv 0 \pmod{q}$ and $\varepsilon(j) = 0$ otherwise.

As for $\operatorname{Re} s > 1$ we have

$$\sum_{p \equiv j \pmod{q}} p^{-s} = (q-1)^{-1} \log(1/(s-1)) + g_{q,j}(s),$$

where the functions $g_{q,j}(s)$ are regular for $\operatorname{Re} s \geq 1$, we get, taking into account the evident equality $\beta(q) = \varepsilon(1) + \dots + \varepsilon(q-1)$, the equation

$$(1) \quad \sum_{n=1}^{\infty} a_0(n) n^{-s} = H_q(s) (s-1)^{-\beta(q)/(q-1)}$$

with

$$(2) \quad H_q(s) = \prod_p \{ (1 + \Phi_0(p, s)) \exp(-a_0(p) p^{-s}) \} \cdot \exp \left\{ a_0(q) q^{-s} + \sum_{j=1}^{q-1} \varepsilon(j) g_{q,j}(s) \right\},$$

which is clearly regular for $\operatorname{Re} s \geq 1$, and not vanishing at $s = 1$.

If now $\beta(q) \neq 0$, then it results by Lemma 1 that

$$N(n \leq x: q \nmid f(n)) = \sum_{n \leq x} a_0(n) = H(1) \Gamma^{-1}(a) x (\log x)^{a-1}$$

with $a = \beta(q)/(q-1)$ and so Theorem I is proved in the case of $m = 0$, $\beta(q) \neq 0$ with $M = M(q, 0) = 0$.

If $\beta(q) = 0$, then $f(n)$ is not divisible by q only if either n is 2-full (i.e. $p|n$ implies $p^2|n$ for every prime p), or $n = qn_1$ and n_1 is 2-full. This implies

$$N\{n \leq x | q \nmid f(n)\} \leq N\{n \leq x | n \text{ 2-full}\} + N\{n \leq xq^{-1} | n \text{ 2-full}\} = O(x^{1/2})$$

by the result of Erdős and Szekeres [4], who proved that $N(n \leq x: n \text{ 2-full}) \sim \zeta(3/2) \cdot \zeta^{-1}(3) \cdot x^{1/2}$.

Finally, note that for $q \in R_f$ (i.e. $\beta(q) = q-1$) we have

$$\sum_{n=1}^{\infty} a_0(n) n^{-s} = \zeta(s) \prod_p (1 + \Phi_0(p, s)) (1 - p^{-s})$$

and the product $\prod_p (1 + \Phi_0(p, s)) (1 - p^{-s})$ is regular for $\operatorname{Re} s \geq 1$. As $\lim_{s \rightarrow 1} \zeta(s)(s-1) = 1$, we get

$$N(n \leq x: q \nmid f(n)) \sim \prod_p (1 + \Phi_0(p, 1)) (1 - p^{-1}) x$$

by the theorem of Ikehara (lemma 1, (i), $a = 1$, $m = 0$), and so the constant $C_3(q, 0)$ is equal to

$$(3) \quad \prod_p (1 + \Phi_0(p, 1)) (1 - p^{-1}).$$

We have thus proved all assertions of Theorem I concerning the case $m = 0$. Now let us turn to the case of $m \geq 1$.

For $i = 1, 2, \dots$, let $t_i(q)$ be the number of integers x ($1 \leq x < q^{i+1}$) not divisible by q and such that $q^i \parallel W(x)$. Let $\Lambda = \Lambda(q)$ be the set of all natural numbers λ with $t_\lambda(q) \neq 0$. We prove now

LEMMA 3. Let $R = (r_1, \dots, r_k) \in \mathcal{R}(m)$, $r_1, \dots, r_N \in \Lambda(q)$, $r_{N+1}, \dots, r_k \notin \Lambda(q)$. Then

$$S_R(s) = \sum_{\langle p_1, \dots, p_k \rangle}^* \prod_{j=1}^k \Phi_{r_j}(p_j, s) (1 + \Phi_0(p_j, s))^{-1}$$

is a polynomial $V_R(t)$ in $t = \log(1/(s-1))$ over the ring Ω of functions regular in the closed half-plane $\operatorname{Re} s \geq 1$.

If the sum (which should be treated as 1 in the case it is void)

$$\sum_{\langle p_{1+N}, \dots, p_k \rangle}^* \prod_{j=1}^k \Phi_{r_j}(p_j, s) (1 + \Phi_0(p_j, s))^{-1}$$

is identically zero, then $V_R(t) = 0$. Otherwise

$$V_R(t) = \sum_{i=0}^N A_i^{(R)}(s) t^i$$

with $A_i^{(R)} \in \Omega$,

$$A_N^{(R)}(s) = \left\{ \prod_{i=1}^N t_{r_i}(q) q^{-1} (q^{1+r_i}) \right\} \sum_{\langle p_{1+N}, \dots, p_k \rangle}^* \prod_{j=1+N}^k \Phi_{r_j}(p_j, s) (1 + \Phi_0(p_j, s))^{-1}$$

and $A_N(1) > 0$.

Proof. Note first that $1 + \Phi_0(p, s)$ does not vanish for $\operatorname{Re} s \geq 1$. Indeed, for $p \neq 2$ we have $|1 + \Phi_0(p, s)| \geq 1/2$ in the closed half-plane $\operatorname{Re} s \geq 1$, and if $\Phi_0(2, s) = -1$, then

$$1 = \left| \sum_{j=1}^{\infty} a_0(2^j) 2^{-js} \right| \leq \sum_{j=1}^{\infty} a_0(2^j) 2^{-j} \leq \sum_{j=1}^{\infty} 2^{-j} = 1$$

which implies $a_0(2^j) = 1$ for $j = 1, 2, \dots$ and, consequently, $1 + \Phi_0(2, s) = (1 - 2^{-s})^{-1} \neq 0$ for $\operatorname{Re} s \geq 1$.

Let us write for shortness

$$\varphi_i(p, s) = \Phi_i(p, s) (1 + \Phi_0(p, s))^{-1}.$$

Note that

$$\begin{aligned} (4) \quad \sum_p \varphi_i(p, s) &= \sum_p \Phi_i(p, s) - \sum_p \Phi_i(p, s) \Phi_0(p, s) (1 + \Phi_0(p, s))^{-1} \\ &= \sum_p a_i(p) p^{-s} + \sum_p \sum_{j=2}^{\infty} a_i(p^j) p^{-js} - \sum_p \Phi_i(p, s) \Phi_0(p, s) (1 + \Phi_0(p, s))^{-1} \\ &= t_i(q) q^{-1} (q^{1+i}) \log(1/(s-1)) + g_i(s), \end{aligned}$$

where $g_i(s) \in \Omega$. In fact, the series $\sum_p \sum_{j=2}^{\infty} a_i(p^j) p^{-js}$ converges uniformly for $\operatorname{Re} s \geq 1/2 + \varepsilon$ ($\varepsilon > 0$), the series

$$\sum_{p \neq 2} \Phi_i(p, s) \Phi_0(p, s) (1 + \Phi_0(p, s))^{-1}$$

converges uniformly for $\operatorname{Re} s \geq \log 2 / \log 3$, as $|1 + \Phi_0(p, s)| \geq 1 - (3^\sigma - 1)^{-1}$ (where $\sigma = \operatorname{Re} s$) implies

$$|\Phi_i(p, s) \Phi_0(p, s) (1 + \Phi_0(p, s))^{-1}| \leq (3^\sigma - 1)(3^\sigma - 2)^{-1} \cdot (p^\sigma - 1)^{-2}$$

and the series $\sum_{p \neq 2} (p^\sigma - 1)^{-2}$ is convergent uniformly for $\sigma \geq 1/2 + \varepsilon$.

Similarly, for $j \geq 2$ and arbitrary i_1, \dots, i_j we have

$$(5) \quad \sum_p \varphi_{i_1}(p, s) \dots \varphi_{i_j}(p, s) = h_{i_1, \dots, i_j}(s) \in \Omega$$

Let now $\psi_1(p, s), \dots$ be arbitrary functions of the form

$$\psi(p, s) = \varphi_{i_1}(p, s) \dots \varphi_{i_j}(p, s)$$

and let

$$S_M(\psi_1, \dots, \psi_M) = \sum_{\langle p_1, \dots, p_M \rangle}^* \psi_1(p_1, s) \dots \psi_M(p_M, s).$$

Then the following recurrence relation holds:

$$(6) \quad S_M(\psi_1, \dots, \psi_M) = S_1(\psi_M) S_{M-1}(\psi_1, \dots, \psi_{M-1}) - \sum_{j=1}^{M-1} S_{M-1}(\psi_1, \dots, \psi_{j-1}, \psi_j \psi_M, \psi_{j+1}, \dots, \psi_{M-1}).$$

Indeed,

$$\begin{aligned} S_M(\psi_1, \dots, \psi_M) &= \sum_{\langle p_1, \dots, p_{M-1} \rangle}^* \psi_1(p_1, s) \dots \psi_{M-1}(p_{M-1}, s) \sum_{\substack{p_M \neq \\ \neq p_1, \dots, p_{M-1}}} \psi_M(p_M, s) \\ &= \sum_{\langle p_1, \dots, p_{M-1} \rangle}^* \psi_1(p_1, s) \dots \psi_{M-1}(p_{M-1}, s) \left\{ \sum_{p_M} \psi_M(p_M, s) - \sum_{j=1}^{M-1} \psi_M(p_j, s) \right\} \\ &= S_1(\psi_M) S_{M-1}(\psi_1, \dots, \psi_{M-1}) - \sum_{j=1}^{M-1} S_{M-1}(\psi_1, \dots, \psi_{j-1}, \psi_j \psi_M, \psi_{j+1}, \dots, \psi_{M-1}). \end{aligned}$$

This recurrent formula implies jointly with (4) and (5) that $S_M(\psi_1, \dots, \psi_M)$ is a polynomial in $\log(1/(s-1))$ over Ω , and the degree of this polynomial does not exceed the number of functions ψ_1, \dots, ψ_M which are equal to one of the functions φ_i with $i \in A$.

Now observe that for $j \leq N$ we have

$$(7) \quad S_j(\varphi_{r_1}, \dots, \varphi_{r_j}) = \left(\prod_{i=1}^j t_{r_i}(q) \varphi^{-1}(q^{1+r_i}) \right) \log^j(1/(s-1)) + U_j(\log(1/(s-1))),$$

where $U_j(t)$ is a polynomial over Ω whose degree does not exceed $j-1$. In fact, for $j=1$, (7) coincides with (4) and the inductive step can be carried out easily with the use of (4) and (6).

Now

$$\begin{aligned} (8) \quad S_R = S_R(\varphi_{r_1}, \dots, \varphi_{r_k}) &= \sum_{\langle p_1, \dots, p_N \rangle}^* \prod_{i=1}^N \varphi_{r_i}(p_i, s) \sum_{\substack{\langle p_{N+1}, \dots, p_k \rangle \\ p_{N+1}, \dots, p_k \neq p_1, \dots, p_N}}^* \prod_{i=N+1}^k \varphi_{r_i}(p_i, s) \\ &= \sum_{\langle p_1, \dots, p_N \rangle}^* \prod_{i=1}^N \varphi_{r_i}(p_i, s) \sum_{\langle p_{N+1}, \dots, p_k \rangle}^* \prod_{i=1+N}^k \varphi_{r_i}(p_i, s) - \\ &\quad - \sum_{\langle p_1, \dots, p_N \rangle}^* \prod_{i=1}^N \varphi_{r_i}(p_i, s) \sum_{\substack{\langle p_{N+1}, \dots, p_k \rangle \\ \{p_{N+1}, \dots, p_k\} \cap \{p_1, \dots, p_N\} \neq \emptyset}}^* \prod_{i=1+N}^k \varphi_{r_i}(p_i, s) = A - B. \end{aligned}$$

Note that $S_{k-N}(\varphi_{r_{N+1}}, \dots, \varphi_{r_k}) \in \Omega$ and so using (7) (for $j=N$) we find that the term A in (8) (which is clearly equal to $S_N(\varphi_{r_1}, \dots, \varphi_{r_N}) \times S_{k-N}(\varphi_{r_{N+1}}, \dots, \varphi_{r_k})$) is a polynomial in $\log(1/(s-1))$ over Ω of the degree N and with the leading coefficient equal to

$$(9) \quad \prod_{i=1}^N (t_{r_i}(q) \varphi^{-1}(q^{1+r_i})) \cdot \sum_{\langle p_{1+N}, \dots, p_k \rangle}^* \prod_{i=1+N}^k \varphi_{r_i}(p_i, s),$$

except the case, when

$$\sum_{\langle p_{1+N}, \dots, p_k \rangle}^* \prod_{i=1+N}^k \varphi_{r_i}(p_i, s)$$

vanishes identically, in which case $A=0$. As $\varphi_{r_i}(p_i, 1) \geq 0$, we find that

$$\prod_{i=1+N}^k \Phi_{r_i}(p_i, 1) \Phi_0(p_i, 1) (1 + \Phi_0(p_i, 1))^{-1} = 0,$$

and so

$$\prod_{i=1+N}^k \Phi_{r_i}(p_i, 1) \Phi_0(p_i, 1) = 0.$$

But

$$\prod_{i=1+N}^k \Phi_{r_i}(p_i, s) \Phi_0(p_i, s)$$

is a Dirichlet series with non-negative coefficients convergent at $s = 1$, and so it can vanish there only if it vanishes identically. This implies the vanishing of S_R and so proves the first part of our lemma.

Now assume that S_R does not vanish identically. The Lemma will be proved if we show that the term B in (8) is a polynomial in $\log(1/(s-1))$ over Ω with degree not exceeding $N-1$.

Consider the set X of all functions $x(m)$ defined on the set $\{N+1, N+2, \dots, k\}$ with values in $\{1, \dots, k\}$ such that for every $m = N+1, \dots, k$ either $1 \leq x(m) \leq N$ or $x(m) = m$, and the set $Z_x = x^{-1}(\{1, 2, \dots, N\})$ is not void. Then

$$B = \sum_{x \in X} S_{N+t}(\psi_1, \dots, \psi_{N+t}),$$

where $t = k - N - |Z_x|$ and where

$$\psi_i(p, s) = \varphi_{r_i}(p, s) \prod_{\substack{j > N \\ x(j)=i}} \varphi_{r_j}(p, s)$$

for $i = 1, 2, \dots, N$ and

$$\psi_i(p, s) = \varphi_{r_j}(p, s)$$

for $i = N+1, \dots, k$, where j is the i -th largest element of the set $\{N+1, \dots, k\} \setminus Z_x$ ($|Y|$ denotes the number of elements of the set Y). It follows from the previous remarks that B is a polynomial over Ω in $\log(1/(s-1))$ of the degree not exceeding

$$\max_{x \in X} |\{1, 2, \dots, N\} \setminus \{x(N+1), \dots, x(k)\}| \leq N-1,$$

as Z_x is not void. The Lemma is thus proved.

Proof of Theorem I. From (1) and the Lemmas 2 and 3 it follows that

$$(10) \quad \sum_{n=1}^{\infty} a_m(n) n^{-s} = H_q(s) (s-1)^{-\beta(q)/(q-1)} \sum_{R \in \mathcal{R}(m)} F(R) V_R(\log(1/(s-1))).$$

If $V_R(t)$ vanishes identically for all R in $\mathcal{R}(m)$, then $a_m(n) = 0$ for all n , and so the set $\{n: q^m \parallel f(n)\}$ is void. Assume thus that there exist systems R in $\mathcal{R}(m)$ such that $V_R(t) \neq 0$. Let $M(q, m)$ be the degree of the polynomial

$$\sum_{R \in \mathcal{R}(m)} F(R) V_R(\log(1/(s-1))).$$

By the last lemma the leading coefficient of this polynomial does not vanish at $s = 1$, and so the cases (i), (ii) and (iv) of Theorem I immediately follow by Lemma 1. (Note that in the case (iv) we have $\beta(q) = q-1$ and the set $\mathcal{A}(q)$ is void, whence $M(q, m) = 0$). In the case (iii)

we have $\beta(q) = 0$ and $M(q, m) = 0$; thus $t_j(q) = 0$ for all j not exceeding m , and so we have $q^{1+m}|f(n)$ for every prime $p \neq q$. It follows that $q^m||f(n)$ can hold only if either n is 2-full or n is divisible by q and n/q is 2-full.

The evaluation

$$N(n \leq x: q^m||f(n)) = O(x^{1/2})$$

follows now in the same way as in the case of $m = 0$, which was treated earlier. Theorem I is thus proved in all cases.

The number $M(q, m)$ can be evaluated explicitly. For every $R = (r_1, \dots, r_k) \in \mathcal{R}(m)$, let $N(R)$ be the number of the r_i 's belonging to $A(q)$. Then Lemma 3 implies that

$$(11) \quad M(q, m) = \max_{\substack{R \in \mathcal{R}(m) \\ V_R(t) \neq 0}} N(R).$$

If $t_1(q) \neq 0$, then $M(q, m) = m$. Indeed, the system $R_0 = (1, \dots, 1)$ belongs to $\mathcal{R}(m)$ and $N(R_0) = m$, whence the equation $M(q, m) = m$ results from the observation that $N(R)$ does not exceed $m-1$ for the remaining systems R in $\mathcal{R}(m)$. Moreover, we have $F(R_0) = (m!)^{-1}$, thus in this case we get

$$C_1(q, m) = (m!)^{-1} t_1^m(q) q^{-m} (q-1)^{-m} C_1(q, 0)$$

and

$$C_2(q, m) = ((m-1)!)^{-1} t_1^m(q) q^{-m} (q-1)^{-m} \prod_p (1 + \Phi_0(p, 1)).$$

To evaluate $t_1(q)$, the following fact may be useful. Let x_1, \dots, x_r be solutions of $W(x) \equiv 0 \pmod{q}$, not divisible by q , and for such x_i 's put

$$s(x_i) = \begin{cases} q & \text{if } W'(x_i) \equiv 0 \pmod{q}, W(x_i) \not\equiv 0 \pmod{q^2}, \\ q-1 & \text{if } W'(x_i) \not\equiv 0 \pmod{q}, \\ 0 & \text{if } W'(x_i) \equiv 0 \pmod{q}, W(x_i) \equiv 0 \pmod{q^2}. \end{cases}$$

Then by theorem 123 of [5] we get $t_1(q) = \sum_{i=1}^r s(x_i)$.

The second case, when it is easy to find $M(q, m)$, arises when, for infinitely many primes p , the function $\Phi_1(p, s)$ does not vanish identically. If $\lambda_0 = \inf \lambda \geq 1$, then $M(q, m) = [m/\lambda_0] = \nu$, say. Indeed, the system $R_1 = (\underbrace{\lambda_0, \dots, \lambda_0}_{\nu \text{ times}}, 1, \dots, 1)$ belongs to $\mathcal{R}(m)$, $N(R_1) = \nu$ and $N(R) < \nu$ for other systems $R \in \mathcal{R}(m)$. As

$$\sum_{\langle p_1, \dots, p_{m-\nu\lambda_0} \rangle}^* \Phi_1(p_1, s) \dots \Phi_1(p_{m-\nu\lambda_0}, s)$$

does not vanish identically, the equality $M(q, m) = \nu$ results.

4. In this section, we prove Theorem II. First, we give an auxiliary result, which is an easy consequence of Ikehara's theorem (lemma 1, (i), $m = 0$).

LEMMA 4. If $a_n \geq 0$, $c > 0$ and $\sum_{n=1}^{\infty} a_n n^{-s}$ is regular for $\operatorname{Re} s \geq c$, then $\sum_{n \leq x} a_n = o(x^{c-\varepsilon})$ with a suitable positive ε .

Proof. Let b be an arbitrary positive number at which the function $\sum_{n=1}^{\infty} a_n n^{-s}$ is regular. Consider the function

$$\sum_{n=1}^{\infty} (a_n + n^{b-1}) n^{-s} = \sum_{n=1}^{\infty} a_n n^{-s} + \zeta(s+1-b) = (s-b)^{-1} + g(s),$$

where $g(s)$ is regular for $\operatorname{Re} s \geq b$.

By Lemma 1, (i) we get

$$\sum_{n \leq x} (a_n + n^{b-1}) \sim x^b/b,$$

and, as obviously $\sum_{n \leq x} n^{b-1} \sim x^b/b$, it results that

$$(12) \quad \sum_{n \leq x} a_n = o(x^b).$$

But the regularity of $\sum_{n=1}^{\infty} a_n n^{-s}$ at $s = c$ implies convergence of this series and the regularity of the sum for some $b < c$, as otherwise the abscissa of convergence for $\sum_{n=1}^{\infty} a_n n^{-s}$ would be equal to c , and the sum would have a singularity at c contrary to the assumption. Lemma 4 now follows from (12).

Now we prove that one of the possibilities (a), (b), (c) stated in Theorem II must hold. Let $D = q_1 \dots q_k$ be the factorization of $D > 1$ into primes, let $\varepsilon_D(n) = 1$ if $D \nmid f(n)$ and $\varepsilon_D(n) = 0$ otherwise, finally let $\varepsilon_q(n) = 1$ if $q \nmid f(n)$ and $\varepsilon_q(n) = 0$ otherwise. Clearly

$$\varepsilon_D(n) = 1 - \prod_{i=1}^k (1 - \varepsilon_{q_i}(n))$$

and so with $\varrho_d(n) = \prod_{q|d} \varepsilon_q(n)$ we have

$$(13) \quad \sum_{n=1}^{\infty} \varepsilon_D(n) n^{-s} = - \sum_{\substack{d|D \\ d \neq 1}} \mu(d) \sum_{n=1}^{\infty} \varrho_d(n) n^{-s}$$

for $\operatorname{Re} s > 1$.

Evidently, $\varrho_d(n) = 1$ if and only if $(d, f(n)) = 1$, thus the functions $\varrho_d(n)$ are multiplicative and we have

$$(14) \quad \sum_{n=1}^{\infty} \varrho_d(n) n^{-s} = \prod_p \left(1 + \sum_{j=1}^{\infty} \varrho_d(p^j) p^{-js} \right) = \exp \left(\sum_{p \nmid d} \varrho_d(p) p^{-s} + g_d(s) \right)$$

with $g_d(s) \in \Omega$.

If $d = p_1 \dots p_t$, then as $W(x) \not\equiv 0 \pmod{p_i}$ holds for $\beta(p_i)$ residue classes $x \pmod{p_i}$ relatively prime to p_i , we find that $(f(p), d) = 1$ holds for all primes p belonging to one of $T(d) = \beta(p_1) \dots \beta(p_t)$ residue classes \pmod{d} relatively prime to d . Thus

$$\sum_{p \nmid d} \varrho_d(p) p^{-s} = T(d) \varphi^{-1}(d) \log(1/(s-1)) + G_d(s)$$

(where $G_d(s) \in \Omega$) and finally by (14) we get

$$\sum_{n=1}^{\infty} \varrho_d(n) n^{-s} = h_d(s) (s-1)^{-T(d) \varphi^{-1}(d)}$$

with $h_d(s) \in \Omega$ and $h_d(1) \neq 0$. Now (13) implies

$$\sum_{n=1}^{\infty} \varepsilon_D(n) n^{-s} = - \sum_{\substack{d|D \\ d \neq 1}} \mu(d) h_d(s) (s-1)^{-T(d) \varphi^{-1}(d)}$$

and we may write

$$\sum_{n=1}^{\infty} \varepsilon_D(n) n^{-s} = \sum_j K_j(s) (s-1)^{-\beta_j} + K(s),$$

where the β_j 's are distinct real numbers, none of which is equal to zero or a negative integer, $K_j(s)$, $K(s)$ are in Ω and $K_j(1) \neq 0$. If the sum

$$\sum_j K_j(s) (s-1)^{-\beta_j}$$

is not void, and $\beta = \max \beta_j$, then Lemma 1 implies

$$\sum_{n \leq x} \varepsilon_D(n) \sim C(D) x (\log x)^{\beta-1}$$

and otherwise

$$\sum_{n=1}^{\infty} \varepsilon_D(n) n^{-s} = K(s) \in \Omega$$

and by Lemma 4 we get $\sum_{n \leq x} \varepsilon_D(n) = o(x^{1-\varepsilon})$ with a suitable positive ε .

We have thus proved that one of the possibilities (a), (b) or (c) of Theorem II must hold.

Now observe that if D has a prime divisor q in R_f , then by Theorem I, (iv), we have $N(n \leq x: q \nmid f(n)) \sim Cx$ with some positive C , as the set $\{n: q \nmid f(n)\}$ contains $n = 1$, and so is not void. Consequently, $N_D(x) \geq C_1 x$ with some $C_1 > 0$, which excludes (a) and (b) and leaves us with (c). If, however, D has no prime divisors from R_f , then by the Corollary we have $N_D(x) = o(x)$, and thus (c) is excluded, and so (a) or (b) must hold. The theorem is thus proved in all cases.

We conclude with some examples. Let first $f(n) = d_k(n)$ be the number of solutions of $x_1 \dots x_k = n$. Then for a prime $q \geq 1 + k$ we easily get by (3) the relation

$$(16) \quad N(n \leq x: q \nmid d_k(n)) \sim (\zeta(q)/\zeta(q+1-k))x.$$

(In the case of $k = 2$ this was obtained by L. G. Sathe).

For $f(n) = \varphi(n)$ we get

$$(17) \quad N\{n \leq x: 2^m \parallel \varphi(n)\} \sim ((m-1)!)^{-1} \cdot 3 \cdot 2^{-1-m} x (\log \log x)^{m-1} (\log x)^{-1}$$

for $q = 2$ and $m \geq 1$, and

$$(18) \quad N\{n \leq x: q^m \parallel \varphi(n)\} \sim C_1(q, 0)(m!)^{-1} q^{-m} x (\log \log x)^m (\log x)^{-1/(q-1)}.$$

for primes $q \neq 2$ (this was proved in [10] by Scourfield).

For the Jordan's function $J_2(n) = n^2 \prod_{p|n} (1 - p^{-2})$ we get

$$(19) \quad N\{n \leq x: 3^m \parallel J_2(n)\} \sim ((m-1)!)^{-1} \cdot 2 \cdot (2/3)^{1+m} x (\log \log x)^{m-1} (\log x)^{-1}$$

for $m \geq 1$, and

$$(20) \quad N(n \leq x: q^m \parallel J_2(n)) \sim C_1(q, 0)(m!)^{-1} (2/q)^m x (\log \log x)^m (\log x)^{-2/(q-1)}$$

for $q \neq 2, 3$.

REFERENCES

- [1] E. Cohen, *Arithmetical Notes IV, A set of integers related to the divisor function*, Journal of the Tennessee Academy of Sciences 37 (1962), p. 119-120.
- [2] — *Arithmetical Notes V, A divisibility property of the divisor function*, American Journal of Mathematics 83 (1961), p. 693-697.
- [3] H. Delange, *Généralisation du théorème de Ikehara*, Annales Scientifiques de l'Ecole Normale Supérieure 71 (1954), p. 213-242.
- [4] P. Erdős und G. Szekeres, *Über die Anzahl der abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem*, Acta Scientiarum Mathematicarum, Szeged, 7 (1934), p. 95-102.
- [5] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4-th ed., Oxford 1962.
- [6] R. A. Rankin, *The divisibility of divisor function*, Proceedings of the Glasgow Mathematical Association 5 (1961), p. 35-40.
- [7] L. G. Sathe, *On a congruence property of the divisor function I*, Journal of the Indian Mathematical Society 7 (1943), p. 143-145.

[8] — *On a congruence property of the divisor function II*, ibidem 7 (1943), p. 146-152.

[9] — *On a congruence property of the divisor function*, American Journal of Mathematics 67 (1945), p. 397-406.

[10] E. J. Scourfield, *On the divisibility of $\sigma_v(n)$* , Acta Arithmetica 10 (1964), p. 245-288.

[11] G. N. Watson, *Über Ramanujansche Kongruenzeigenschaften der Zerfallungsanzahlen I*, Mathematische Zeitschrift 39 (1935), p. 712-731.

[12] E. Wirsing, *Das asymptotische Verhalten von Summen über multiplikative Funktionen*, Mathematische Annalen 143 (1961), p. 75-102.

INSTITUTE OF MATHEMATICS OF THE WROCLAW UNIVERSITY

Reçu par la Rédaction le 19. 2. 1966
