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FRATTINI SUBLATTICES IN VARIETIES OF LATTICES

BY

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1. Introduction. The *Frattini sublattice of a lattice* L, denoted by $\Phi(L)$, is the intersection of all the maximal proper sublattices of L. In [4], Koh showed that for every lattice L there exists a lattice L^+ such that $L \cong \Phi(L^+)$. In [1], a topological construction was given that showed that, in the case L is a distributive lattice, L^+ may also be chosen to be distributive. Since a sublattice is allowed to be empty, a representation of the above form is not possible in all varieties of lattices. However, we* will prove the following

THEOREM. *Let V be a nontrivial variety of lattices. If the lattice L is a member of V, then there exists a lattice L^+ of V such that $L \cong \Phi(L^+)$.*

For further information and references, see [3].

2. Proof of the Theorem. Let L be a lattice in a nontrivial variety V of lattices. For $1 \leqslant i < \omega_0$, let A_i denote the lattice of all finite subsets of L ordered by inclusion; for $l \in L$, set $l_i = \{l\} \in A_i$. Let P be the weak direct product of $(L_i : i < \omega_0)$, where $L_0 = L$ and, for $0 < i < \omega_0$, $L_i = A_i$, that is the sublattice of the direct product whose elements are zero on all but finitely many components. For $p \in P$, let $p(i)$ denote the i-th projection. In order to define L^+ we need a lemma. This lemma also provides the means to describe various sublattices of P and, subsequently, of L^+.

LEMMA 1. *Let $j < \omega_0$, $L' \subseteq L$, and $I' \subseteq \omega_0 \setminus 0$. Further, let F_l be a filter of L_j for $l \in L'$. Set P' to be such elements $p \in P$ that if $p(i) \geqslant l_i$ for some $i \in I'$ and $l \in L'$, then $p(j) \in F_l$. Then P' is a sublattice of P.*

Proof. Let $p_1, p_2 \in P'$. If $(p_1 \vee p_2)(i) \geqslant l_i$ for some $i \in I'$ and $l \in L'$, then $p_1(i) \vee p_2(i) \geqslant l_i$ and, by distributivity, $p_k(i) \geqslant l_i$ for $k = 1$ or 2. Thus $p_k(j) \in F_l$, and hence

$$p_1(j) \vee p_2(j) = (p_1 \vee p_2)(j) \in F_l;$$

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that is to say that $p_1 \vee p_2 \in P'$. Similarly, if $(p_1 \wedge p_2)(i) \geqslant l_i$ for some $i \in I'$ and $l \in L'$, then $p_k(i) \geqslant l_i$ for $k = 1$ and 2. Consequently, $p_k(j) \in F_l$ for $k = 1$ and 2. Hence

$$p_1(j) \wedge p_2(j) = (p_1 \wedge p_2)(j) \in F_l;$$

thus $p_1 \wedge p_2 \in P'$.

Using Lemma 1, we now define L^+ as a sublattice of P. Let $j = 0$, $L' = L$, $I' = \omega_0 \setminus 0$, and, for $l \in L'$, let $F_l = [l) \subseteq L_0$. Thus L^+ is the sublattice of P such that, for $p \in P$, if $p(i) \geqslant l_i$ for some i $(1 \leqslant i < \omega_0)$, then $p(0) \geqslant l$. Observe that since L_i for $1 \leqslant i < \omega_0$ is distributive, we infer that if L is a member of a nontrivial variety V, then so is P. Thus L^+ is also in V since it is a sublattice of P.

We now begin to investigate the maximal proper sublattices of L^+.

Again by Lemma 1, for $l \in L$ and distinct i, j $(1 \leqslant i, j < \omega_0)$, the elements $p \in P$ such that if $p(i) \geqslant l_i$, then $p(j) \geqslant l_j$, form a sublattice of P; in this application, $L' = \{l\}$, $I' = \{i\}$, and $F_l = [l_j) \subseteq L_j$. Let L_{lij} denote the intersection of the above-defined sublattice of P with L^+.

LEMMA 2. *For $l \in L$ and distinct i, j $(1 \leqslant i, j < \omega_0)$, L_{lij} is a maximal proper sublattice of L^+.*

Proof. Consider $p_0 \in P$ defined by $p_0(0) = l$, $p_0(i) = l_i$, and $p_0(k) = 0$ for $k \in \omega_0 \setminus \{0, i\}$. Thus $p_0 \in L^+$, but $p_0 \notin L_{lij}$. Hence L_{lij} is a proper sublattice of L^+.

Let $p \in L^+ \setminus L_{lij}$; thus, in particular, $p(i) \geqslant l_i$ (whence $p(0) \geqslant l$) and $p(j) \not\geqslant l_j$. It is required to show that $[L_{lij} \cup \{p\}]$, the sublattice generated by L_{lij} and p, is actually equal to L^+. Choose $q \in L^+ \setminus (L_{lij} \cup \{p\})$; hence $q(i) \geqslant l_i$ (thus $q(0) \geqslant l$) and $q(j) \not\geqslant l_j$. Let us define $p_1 \in L^+$ by $p_1(0) = l$, $p_1(i) = l_i$, $p_1(j) = l_j$, and $p_1(k) = 0$ for $k \neq 0, i, j$; thus $p_1 \in L_{lij}$. Consequently,

$$p_2 = p \wedge p_1 \in [L_{lij} \cup \{p\}].$$

Observe that $p_2(k) = 0$ for $k \neq 0, i, j$, $p_2(0) = l$, $p_2(i) = l_i$, and $p_2(j) = 0$ (since $p(j) \not\geqslant l_j$ and l_j is an atom of A_j). Further, $p_3 \in L^+$ is defined by $p_3(i) = q(i) \setminus l_i$ and, for $k \neq i$, $p_3(k) = q(k)$ is an element of L_{lij}. Hence

$$q = p_2 \vee p_3 \in [L_{lij} \cup \{p\}].$$

We conclude that L_{lij} is a maximal sublattice of L^+.

For $p \in L^+$, if $p(k) = 0$ for all k $(1 \leqslant k < \omega_0)$, then $p \in L_{lij}$ for any $l \in L$ and $1 \leqslant i, j < \omega_0$. Alternately, if $p(i) \neq 0$ for some i $(1 \leqslant i < \omega)$, then $l \in p(i)$ for some $l \in L$. Further, $p(j) = 0$ for some j $(1 \leqslant j < \omega_0)$. Thus $p \notin L_{lij}$. We deduce

LEMMA 3. *Let $\Lambda = \bigcap (L_{lij}: l \in L$ and $i \neq j$ for $1 \leqslant i, j < \omega_0)$. Then*

$$\Phi(L^+) \subseteq \Lambda \cong L_0 = L.$$

Thus, if $p \in \Phi(L^+)$, then $p(j) = 0$ for $1 \leqslant j < \omega_0$. The proof of the Theorem will be complete when we have shown that if, for some $p \in L^+$, $p(j) = 0$ for all j $(1 \leqslant j < \omega_0)$, then $p \in \Phi(L^+)$; that is to say that any such p is an element of every maximal proper sublattice.

LEMMA 4. *Let M be a maximal proper sublattice of L^+. Then for every $l \in L$ there exists $p \in M$ such that $p(0) = l$.*

Proof. Suppose, to the contrary, that there exists $l \in L$ such that $p(0) \neq l$ for all $p \in M$. Set

$$F = [\{p(0)\colon\, p \in M \text{ and } p(1) \geqslant l_1\}).$$

Thus F is a filter of $L_0 = L$. Moreover, for $p \in M \subseteq L^+$, if $p(1) \geqslant l_1$, then $p(0) \geqslant l$. That is to say, $F \subseteq [l)$. However, $f \in F$ if and only if $f \geqslant p_1(0) \wedge \ldots \wedge p_n(0)$ for $p_i \in M$ $(1 \leqslant i \leqslant n)$; in particular, $l \in F$ if and only if $l \geqslant p_1(0) \wedge \ldots \wedge p_n(0)$ for $p_i \in M$ $(1 \leqslant i \leqslant n)$. Since $p_1 \wedge \ldots \wedge p_n \in M$, this is impossible. Hence $F \subseteq [l)$.

Let $j = 0, L' = \{l\}$, and $I' = \{1\}$. Furthermore, let $F_l = F$. By Lemma 1, the set P' of elements of P such that if $p(1) \geqslant l_1$, then $p(0) \in F$, defines a sublattice of P. Let $M' = L^+ \cap P'$. Define $q_1 \in L^+$ by $q_1(0) = l$, $q_1(1) = l_1$, and $q_1(i) = 0$ for $2 \leqslant i < \omega_0$. Since $q_1 \notin M'$, M' is a proper sublattice of L^+. However, by construction, $M \subseteq M'$. Thus $M = M'$. Let $q_2 \in L^+$, where $q_2(0) = l$ and $q_2(i) = 0$ for $1 \leqslant i < \omega_0$. By definition, $q_2 \in M'$, and hence $q_2 \in M$. This contradicts the assumption and completes the proof.

LEMMA 5. *Let M be a maximal proper sublattice of L^+. If, for $l \in L$, $p \in L^+$ is the element such that $p(0) = l$ and $p(i) = 0$ for $1 \leqslant i < \omega_0$, then $p \in M$.*

Proof. Let $l \in L$; we will show that $p \in M$. By Lemma 4, there exists $q_1 \in M$ such that $q_1(0) = l$. By Lemma 1, for $l, l' \in L$ and distinct i, j $(1 \leqslant i, j < \omega_0)$, the elements $q \in L^+$ such that if $q(i) \geqslant l_i$, then $q(j) \geqslant l'_j$, form a proper sublattice; let $L_{ll'ij}$ denote this sublattice. Consider the element $p_1 \in L^+$ defined by $p_1(0) = l$, $p_1(i') = l_{i'}$, and $p_1(k) = 0$ for $k \neq 0, i'$. If $i' \neq i$, then $p_1 \in L_{ll'ij}$. However, for $l'' \in L$ and distinct i', j' $(1 \leqslant i', j' < \omega_0)$, $p_1 \notin L_{ll''i'j'}$. Let us suppose that there exist $l' \in L$ and distinct i, j $(1 \leqslant i, j < \omega_0)$ such that $M \subseteq L_{ll'ij}$; since M is maximal, we deduce that $M = L_{ll'ij}$. From this observation it follows that, for all $l'' \in L$ and distinct i', j' $(1 \leqslant i', j' < \omega_0)$, if $i' \neq i$, then $M \nsubseteq L_{ll''i'j'}$. Either way we conclude that there exists an i $(1 \leqslant i < \omega_0)$ such that $M \nsubseteq L_{ll'ij}$ for any $l' \in L$ and any j $(1 \leqslant j < \omega_0)$ distinct from i; further we may choose i such that $q_1(i) = 0$. Consequently, for any $l' \in L$ and any j $(1 \leqslant j < \omega_0)$ distinct from i, there exists $q \in M$ such that $q(i) \geqslant l_i$ (hence $q(0) \geqslant l$) and $q(j) \ngeqslant l'_j$. However, for $1 \leqslant j < \omega_0$, $q_1(j)$ is a finite subset of L and is nonzero on only finitely many components. Thus we may systematically choose finitely many elements $q_k \in M$ (one for each $l' \in L$ and $1 \leqslant j < \omega_0$ with $l'_j \subseteq q_1(j)$) such that $q_k(i) \geqslant l_i$ and $q_k(j) \ngeqslant l'_j$. We now simply observe that

$$p = q_1 \wedge q_2 \wedge \ldots \wedge q_n.$$

In conclusion, $\Phi(L^+) \cong L$.

3. Concluding remark. In the construction presented here the cardinality of L^+ is always infinite. In general this is unavoidable; independently, in [1] and [2], it is shown that there are (infinitely many) finite distributive lattices which are not isomorphic to the Frattini sublattices of any finite distributive lattices.

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