

COMPACTNESS AND HOMOMORPHISMS
OF ALGEBRAIC STRUCTURES

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This paper is closely related to Keisler [4] and [5]. First we give here extensions of his results concerning infinitely long formulas for positively compact structures. Then we give a proof of a generalization of Morley-Vaught's isomorphism theorem on saturated structures (Theorem 6) which permits us to derive another result (Theorem 5).

Those theorems give additional informations on positively compact structures. Such structures have been already studied in papers [7], [9] and [10]. Recall that (see e.g. [9]) our positively \mathfrak{m} -compact structures are defined similarly to \mathfrak{m}^+ -saturated structures of Keisler (for structures having at most \mathfrak{m} relations and functions, as is always supposed in this paper, the sole difference is that we replace in the definition arbitrary formulas by positive ones). To make the terminology uniform, following [10], we say "elementarily \mathfrak{m} -compact", where " \mathfrak{m}^+ -saturated" could be used as well.

Our notation and general suppositions are the same as in [4].

1. First we will derive from the results of Keisler ([5], Theorem 5.1 and 5.3) the following theorem:

THEOREM 1. *Let $2^{\mathfrak{m}} = \mathfrak{m}^+$, \mathfrak{A} be an elementarily \mathfrak{m} -compact algebraic structure and \mathfrak{B} a positively compact one, both of power at most \mathfrak{m}^+ . A necessary and sufficient condition for \mathfrak{B} to be a strong homomorphic image of \mathfrak{A} (or isomorphic to a retract of \mathfrak{A}) is that $\text{Th}(\mathfrak{A}) \cap \Gamma \subseteq \text{Th}(\mathfrak{B})$, where Γ is the set of all sentences which are preserved by strong homomorphisms (or, respectively, retractions) ⁽¹⁾.*

Proof. The necessity is obvious.

Assuming that $2^{\mathfrak{m}} = \mathfrak{m}^+$, a theorem of Mycielski [7] yields that there exists an elementarily \mathfrak{m} -compact algebraic structure \mathfrak{C} of power \mathfrak{m}^+ , which is an elementary extension of \mathfrak{B} . Since \mathfrak{B} is positively compact,

⁽¹⁾ For a detailed study of such formulas, see [5].

we see that, by Theorem 2.3 of [10], \mathfrak{B} is a retract of \mathfrak{C} . Finally, applying Theorem 5.1 of [5] (or 5.3 of [5]) to the structures \mathfrak{A} and \mathfrak{C} we see that \mathfrak{C} is a strong homomorphic image (is isomorphic to a retract) of \mathfrak{A} . Thus \mathfrak{B} is a strong homomorphic image (is isomorphic to a retract) of \mathfrak{A} .

Now we give three theorems which are similar to Theorems 2.3, 3.1 and 3.2 of [4].

Let Π denote the set of all positive formulas.

THEOREM 2. *Suppose that \mathfrak{A} is positively \mathfrak{m} -compact. Then for every conjunctive formula φ such that $\mathcal{A}(\varphi) \subseteq \Pi$ we have*

$$\mathfrak{A} \models (\forall v_0 \dots \forall v_\alpha \dots)_{\alpha < \mathfrak{m}^+} (\bigwedge \mathcal{A}(\varphi) \rightarrow \varphi).$$

The proof is the same as the proof of Theorem 2.3 in [4] but uses positive \mathfrak{m} -compactness of \mathfrak{A} instead of elementary \mathfrak{m} -compactness of \mathfrak{A} .

THEOREM 3. *Suppose that \mathfrak{A} is positively \mathfrak{m} -compact. Then for every admissible formula φ such that $\mathcal{A}(\varphi) \subseteq \Pi$ we have*

$$\mathfrak{A} \models (\forall v_0 \dots \forall v_\alpha \dots)_{\alpha < \mathfrak{m}^+} (\bigwedge \mathcal{A}(\varphi) \rightarrow \varphi).$$

The proof of this theorem is the same as the proof of Theorem 3.1 in [4] up to the usage of our Theorem 2 instead of Theorem 2.3 of [4].

Now, let us remark that, in some sense, Theorems 2 and 3 extend Theorem 2.3 of [4] and the first part of Theorem 3.1 of [4], since there are no algebraic structures being \mathfrak{n} -saturated for each cardinal \mathfrak{n} except finite structures, but there are infinite positively compact structures (i.e. structures which are positively \mathfrak{n} -compact for every cardinal \mathfrak{n}). The second part of Theorem 3.1 of [4] can not be extended in this way as the following example of Jan Mycielski shows.

EXAMPLE 1. Let $\mathfrak{A} = \langle A, (R_n)_{n \in \omega} \rangle$ be a structure in which A is the half-closed real interval $(0, 1]$ and $R_n = \{x \in A : x \geq 1/(n+1)\}$. It is easy to see that \mathfrak{A} is atomic compact, whence by Theorem 2.3 of [10], positively compact. Let $\varphi = (\forall x) \bigvee \{R_n(x) : n \in \omega\}$. It is visible that $\mathfrak{A} \models \varphi$, but $\mathfrak{A} \models \neg \bigvee \mathcal{A}(\varphi)$, since each $\vartheta \in \mathcal{A}(\varphi)$ is of the form $(\forall x) (R_{n_0}(x) \vee \dots \vee R_{n_k}(x))$ for $n_i \in \omega$, $i = 0, \dots, k$.

From Theorems 2, 3 and 3.1 of [4], we are going to derive the following theorem, which is analogous to Theorem 3.2 of [4]:

THEOREM 4. *Let \mathfrak{A} be an elementarily \mathfrak{m} -compact structure and \mathfrak{B} a positively \mathfrak{m} -compact one. Let Ψ be the class of all admissible formulas φ such that $\mathcal{A}(\varphi) \subseteq \Pi$ and for all $a \in A^{\mathfrak{m}^+}$ and $b \in B^{\mathfrak{m}^+}$ if $\mathfrak{A} \models \varphi[a]$ and $\mathfrak{B} \models \varphi^*[b]$, then there is a $\vartheta \in \mathcal{A}(\varphi)$ such that $\mathfrak{A} \models \vartheta[a]$ and $\mathfrak{B} \models \neg \vartheta[b]$. Then:*

- (i) all finite positive formulas are in Ψ ;
- (ii) if $\varphi \in \Phi_C \cup \Phi_C^*$ and $\mathcal{A}(\varphi) \subseteq \Pi$, then $\varphi \in \Psi$;
- (iii) if $\Theta \subseteq \Psi$ and Θ is a set, then $\bigvee \Theta \in \Psi$;

(iv) if $\psi \in \Psi$ and (Q_x) is a quantifier of length $a \leq m^+$ and $\varphi = (Q_x)\psi$ has no more than m free variables, then $\varphi \in \Psi$.

Proof. (i) is trivial.

(ii). Let $\varphi \in \Phi_C$ and $\mathcal{A}(\varphi) \subseteq \Pi$. Since $\models \varphi \rightarrow \bigwedge \mathcal{A}(\varphi)$ and $\models \varphi^* \rightarrow \neg \varphi$, we have $\mathcal{U} \models \bigwedge \mathcal{A}(\varphi)[a]$ and $\mathcal{B} \models \neg \varphi[b]$. By Theorem 2, $\mathcal{B} \models \neg \bigwedge \mathcal{A}(\varphi)[b]$, thus there is a $\vartheta \in \mathcal{A}(\varphi)$ such that $\mathcal{B} \models \neg \vartheta[b]$ and $\mathcal{U} \models \vartheta[a]$. Let $\varphi \in \Phi_C^*$. Since $\varphi^* \in \Phi_C$, we have $\mathcal{B} \models \bigwedge \mathcal{A}(\varphi^*)[b]$, by Lemma 2.1, (ii), of [4]. On the other hand, \mathcal{U} is elementarily m -compact, thus, using Theorem 3.1 of [4], we have $\mathcal{U} \models \bigvee \mathcal{A}(\varphi)[a]$. But this means that there is a $\vartheta \in \mathcal{A}(\varphi)$ such that $\mathcal{U} \models \vartheta[a]$. In view of $\varphi^* \in \Phi_C$ and of Lemma 2.1, (i), of [4] we have $\mathcal{U}(\varphi^*) = (\mathcal{U}(\varphi))^*$ and $\neg \vartheta = \vartheta^* \in \mathcal{A}(\varphi^*)$. Hence $\mathcal{B} \models \neg \vartheta[b]$ and (ii) holds.

(iii) It is visible that it suffices to consider two element sets Θ . Let $\psi, \eta \in \Psi$ and $\varphi = \psi \wedge \eta$. Let $\mathcal{U} \models \varphi[a]$ and $\mathcal{B} \models \psi^*[b]$ or $\mathcal{B} \models \eta^*[b]$; say $\mathcal{B} \models \psi^*[b]$. Since $\psi \in \Psi$, there is a $\sigma \in \mathcal{A}(\psi)$ such that $\mathcal{U} \models \sigma[a]$ and $\mathcal{B} \models \neg \sigma[b]$. Since \mathcal{U} is elementarily m -compact, there is, by Theorem 3.1 of [4], a $\tau \in \mathcal{A}(\eta)$ such that $\mathcal{U} \models \tau[a]$. Whence $\mathcal{U} \models (\sigma \wedge \tau)[a]$ and $\mathcal{B} \models \neg (\sigma \wedge \tau)[b]$, but since $(\sigma \wedge \tau) \in \mathcal{A}(\varphi)$, we have $\varphi \in \Psi$, which shows (iii).

The proof of (iv) is similar to Keisler's proof of Theorem 3.2 in [4] but uses positive m -compactness of \mathcal{B} instead of elementary m -compactness of \mathcal{B} .

2. In this section we do not suppose that the symbol of equality is necessarily interpreted as identity. We suppose, however, that the language of our structures, has less than n relations ($n > \aleph_0$ is a fixed cardinal number).

The main result of this paper is the following theorem, which under stronger assumptions (such as in Theorem 1), would follow from Theorem 3.1 of Keisler [5].

THEOREM. 5. *Let \mathcal{U} be a structure which is elementarily p -compact and \mathcal{B} positively p -compact for all $p < n$, where powers of \mathcal{U} and \mathcal{B} are at most n . A necessary and sufficient condition for \mathcal{B} to be a homomorphic image of \mathcal{U} is that $\text{Th}(\mathcal{U}) \cap \Pi \subseteq \text{Th}(\mathcal{B})$.*

For $n = m^+$, we can prove it similarly as Theorem 3.1 in [5] by using our Theorem 4 instead of Theorem 3.2 of [4], but in general this method is not sufficient.

First we give another result. For a positive formula φ let us define formula $\bar{\varphi}$ recursively as follows:

- (i) if φ is atomic, then $\bar{\varphi} = \varphi$;
- (ii) if $\varphi = \psi \wedge \eta$, then $\bar{\varphi} = \bar{\psi} \vee \bar{\eta}$;
- (iii) if $\varphi = \psi \vee \eta$, then $\bar{\varphi} = \bar{\psi} \wedge \bar{\eta}$;
- (iv) if $\varphi = \forall x \psi$, then $\bar{\varphi} = \exists x \bar{\psi}$;
- (v) if $\varphi = \exists x \psi$, then $\bar{\varphi} = \forall x \bar{\psi}$;

Now we can formulate the following theorem, which is a common generalization of Theorem 5, and of Morley-Vaught's theorem on isomorphisms of saturated structures (see e.g. [6]).

THEOREM 6. *Let \mathfrak{A} and \mathfrak{B} be two structures of power at most \mathfrak{n} which are positively \mathfrak{p} -compact for each $\mathfrak{p} < \mathfrak{n}$. Let*

$$\mathfrak{A} \models \varphi \quad \text{or} \quad \mathfrak{B} \models \bar{\varphi}$$

for every (finite) positive sentence φ . Then there are two sequences $a = (a_\alpha)_{\alpha < \mathfrak{n}}$ and $b = (b_\alpha)_{\alpha < \mathfrak{n}}$ such that

- (i) $\{a_\alpha: \alpha < \mathfrak{n}\} = \mathfrak{A}$ and $\{b_\alpha: \alpha < \mathfrak{n}\} = \mathfrak{B}$;
- (ii) $\mathfrak{A} \models \varphi[a]$ or $\mathfrak{B} \models \bar{\varphi}[b]$ for every finite positive formula φ .

The proof of this theorem can be obtained by the method of Cantor [1] of choosing two sequences ⁽²⁾ (see also [2], [3] and [8]).

Proof of Theorem 5. Let us take the language \mathcal{L} having symbols of all relations from \mathfrak{A} and \mathfrak{B} , and interpret it in \mathfrak{B} in the usual way but in \mathfrak{A} put the complements of relations instead of relations, thus e.g. equality in \mathcal{L} is interpreted in \mathfrak{A} as \neq and in \mathfrak{B} as $=$. Then Theorem 5 easily follows from Theorem 6.

THEOREM 7 (Morley-Vaught). *Let \mathfrak{A} and \mathfrak{B} be structures of powers at most \mathfrak{n} which are elementarily \mathfrak{p} -compact for all $\mathfrak{p} < \mathfrak{n}$, and such that $\text{Th}(\mathfrak{A}) = \text{Th}(\mathfrak{B})$. Then \mathfrak{A} and \mathfrak{B} are isomorphic.*

Proof. Let us take a language \mathcal{L} having symbols for all relations of \mathfrak{A} and for all their complements, and interpret it in \mathfrak{A} and \mathfrak{B} similarly as in the proof of Theorem 5. Again Theorem 7 immediately follows from Theorem 6.

Finally we show that the assumption of elementary compactness of \mathfrak{A} in Theorems 1 and 5 is essential.

EXAMPLE 2. Let \mathfrak{Z}_2 and \mathfrak{Z}_3 be the cyclic groups of order 2 and 3, respectively. Let us consider the structures $\mathfrak{A} = \mathfrak{Z}_2^{\aleph_0} \times \mathfrak{Z}_3^{2^{\aleph_0}}$ and $\mathfrak{B} = \mathfrak{Z}_2^{2^{\aleph_0}} \times \mathfrak{Z}_3^{\aleph_0}$. It is easy to see that \mathfrak{A} is elementarily equivalent to \mathfrak{B} and that \mathfrak{A} and \mathfrak{B} are positively compact systems because of their topological compactness and of [10]. In spite of this there is no homomorphism of \mathfrak{A} onto \mathfrak{B} or conversely.

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⁽²⁾ He invented it for showing that all countable dense linear orders without the first and the last elements are isomorphic.

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