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BOOLEAN OPERATIONS OVER MEASURE ALGEBRAS

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We shall investigate Boolean operations over Boolean algebras with a measure. In Sections 1 and 2 we describe operations connected with the forcing theory which applied to measure algebras yield a measure algebra as a result. In the rest of the paper we give many examples which show that usual operations like minimal products and limits do not preserve measurability of an algebra.

The paper was inspired by our Wrocław forcing seminar led by Dr. J. Cichoń.

0. Introduction.

Definition. We say that $\boldsymbol{B}$ is a *measure algebra* iff $\boldsymbol{B}$ is a complete Boolean algebra (c.b.a.) and there is a function $\mu\colon \boldsymbol{B} \to [0, 1]$ such that:

(i) $\mu(\mathbf{1}) = 1$,

(ii) μ is σ-additive, i.e. whenever A is a countable and pairwise disjoint family of elements of $\boldsymbol{B}$ then $\mu(\sum A) = \sum_{a \in A} \mu(a)$,

(iii) μ is strictly positive, i.e. $\mu(a) = 0$ iff $a = \mathbf{0}$.

We call any function $\mu\colon \boldsymbol{B} \to [0, 1]$ satisfying (i), (ii), (iii) a *measure on* $\boldsymbol{B}$.

A typical example: if $(\mathscr{X}, \mathscr{F}, \mu)$ is a measure space with a probability measure μ on a σ-field $\mathscr{F}$ of subsets of $\mathscr{X}$ then the factor algebra $\mathscr{F}/I_\mu$, where $I_\mu = \{A \in \mathscr{F}\colon \mu(A) = 0\}$, is a measure algebra. The measure μ induces a measure on $\mathscr{F}/I_\mu$. We will use the same symbol to denote these two measures. The equivalence classes of elements $A \in \mathscr{F}$ in $\mathscr{F}/I_\mu$ will be denoted by $[A]$.

It follows from the Loomis–Sikorski theorem ([6]) that the above case is in fact general, i.e. for every measure algebra $\boldsymbol{B}$ there exists a measure space $(\mathscr{X}, \mathscr{F}, \mu)$ such that $\boldsymbol{B}$ is isomorphic to $\mathscr{F}/I_\mu$. Particularly, for a measure space $([0, 1], \mathrm{Bor}([0, 1]), \lambda)$, where λ is a Lebesgue measure, $\mathrm{Bor}([0, 1])$ is a set of Borel subsets of $[0, 1]$, we obtain a measure algebra which will be denoted by $\mathscr{R}$.

We assume that the reader is familiar with the theory of Boolean-valued models of set theory ([9], [4], [11]).

Now let us remind the description of reals in $V^{\mathbf{B}}$ for a measure algebra $\mathbf{B}$, due to D. Scott [7].

For a measure space $(\mathcal{X}, \mathcal{F}, \mu)$ such that $\mathbf{B} \cong \mathcal{F}/I_\mu$ we denote by M a family of all measurable functions $f\colon \mathcal{X} \to \mathbf{R}$. If $f \in M$, then f determines an element $f^0 \in V^{\mathbf{B}}$ as follows: we define the Boolean-valued Dedekind cut Q_f,

$$Q_f = \{\langle \check{r}, [\![x\colon f(x) < r]\!]\rangle\colon r \in \mathbf{Q}\},$$

where $\mathbf{Q}$ stands for the set of rational numbers. Clearly,

$V^{\mathbf{B}} \models$ "Q_f is a nonempty set of rational numbers bounded from the right".

Now pick $f^0 \in V^{\mathbf{B}}$ such that

$$V^{\mathbf{B}} \models \text{"}f^0 = \sup Q_f\text{"}.$$

We state some properties of the mapping $f \to f^0$ (see [7]):

(1) If $a, b \in \mathbf{R}$ and $f \in M$ then
$[\![f^0 = \check{a}]\!] = [\![x\colon f(x) = a]\!]$,
$[\![\check{a} < f^0 < \check{b}]\!] = [\![x\colon a < f(x) < b]\!]$.

(2) If $f, g \in M$ then
$[\![f^0 \leqslant g^0]\!] = [\![x\colon f(x) \leqslant g(x)]\!]$, in particular $[\![f^0 = g^0]\!] = \mathbf{1}$ iff $f = g$ almost everywhere.

(3) If $f_1, \ldots, f_n \in M$ then
$[\![f_1^0 + \ldots + f_n^0 = (f_1 + \ldots + f_n)^0]\!] = \mathbf{1}$.

(4) If $t \in V^{\mathbf{B}}$ and $[\![t \in \mathbf{R}]\!] = \mathbf{1}$ then there exists $f \in M$ such that $[\![f^0 = t]\!] = \mathbf{1}$ (by (2), f is determined almost everywhere).

(5) If $f_1, f_2, \ldots$ is a sequence of elements of M, then $[\![\lim_{n\in\omega} f_n^0 = f^0]\!] = \mathbf{1}$ iff $\lim_{n\in\omega} f_n = f$ a.e.

We refer to these properties as to the Scott lemma.

1. Iterations. If $\mathbf{B}$ is c.b.a. and $V^{\mathbf{B}} \models$ "$\mathbf{D}$ is c.b.a" then we define $\mathbf{B}*\mathbf{D}$ to be the *iteration algebra* of $\mathbf{B}$ and $\mathbf{D}$ ([9]). We recall this construction which will be needed in the sequel. Let $\hat{\mathbf{D}} = \{a \in V^{\mathbf{B}}\colon [\![a \in \mathbf{D}]\!] = \mathbf{1}\}$. We can assume that $V^{\mathbf{B}}$ is normalized, i.e. $a = b$ iff $[\![a = b]\!] = \mathbf{1}$.

Now define Boolean operations on $\hat{\mathbf{D}}$:

$$a + b = c \quad \text{iff} \quad [\![a + b = c]\!] = \mathbf{1},$$

$$a \cdot b = c \quad \text{iff} \quad [\![a \cdot b = c]\!] = \mathbf{1}.$$

Let $\mathbf{B}*\mathbf{D}$ be the set $\hat{\mathbf{D}}$ with such defined operations. Denote also by i the canonical embedding from $\mathbf{B}$ into $\mathbf{B}*\mathbf{D}$ given by the equations:

for $b \in \mathbf{B}$

$$[\![i(b) = \mathbf{1}]\!] = b, \quad [\![i(b) = \mathbf{0}]\!] = -b.$$

THEOREM 1. *If $\boldsymbol{B}$ is a measure algebra and $V^{\boldsymbol{B}} \models$ "$\boldsymbol{D}$ is a measure algebra", then $\boldsymbol{B}*\boldsymbol{D}$ is a measure algebra. Moreover, if μ is a measure on $\boldsymbol{B}$ and i is the canonical embedding of $\boldsymbol{B}$ into $\boldsymbol{B}*\boldsymbol{D}$ then we can construct a measure λ on $\boldsymbol{B}*\boldsymbol{D}$ such that for $b \in \boldsymbol{B}$*

$$\lambda(i(b)) = \mu(b).$$

Proof. Let $\boldsymbol{B} = \mathscr{F}/I_\mu$, $v \in V^{\boldsymbol{B}}$ and $V^{\boldsymbol{B}} \models$ "v is a measure on $\boldsymbol{D}$". We define λ on $\boldsymbol{B}*\boldsymbol{D}$ as follows:

If $a \in \boldsymbol{B}*\boldsymbol{D}$ then $[\![a \in \boldsymbol{D}]\!] = \mathbf{1}$. Hence there exists a function $f_a \in M$ such that $[\![v(a) = f_a^0]\!] = \mathbf{1}$.

Let $\lambda(a) = \int f_a \, d\mu$. We are going to show that λ is a measure on $\boldsymbol{B}*\boldsymbol{D}$:

(a) λ is well defined: If $a = b$ then $[\![a = b]\!] = \mathbf{1}$, hence $[\![v(a) = v(b)]\!] = \mathbf{1}$ and $f_a = f_b$ a.e., so $\lambda(a) = \int f_a \, d\mu = \int f_b \, d\mu = \lambda(b)$.

(b) λ is σ-additive: Let $\{a_n : n \in \omega\} \subseteq \boldsymbol{B}*\boldsymbol{D}$ and $a_n \cdot a_m = \mathbf{0}$ for $n \neq m$.

It is proved in [9] that if $[\![a = \sum_{n\in\omega} a_n]\!] = \mathbf{1}$ then $a = \sum_{n\in\omega} a_n$.

Now we pick $g_n \in M$ for $n \in \omega$ such that

$$[\![g_n^0 = v(\sum_{k\leqslant n} a_k)]\!] = \mathbf{1}.$$

Then by the Scott lemma

$$g_n = \sum_{k\leqslant n} f_{a_k} \quad \text{a.e.} \quad \text{and} \quad [\![\lim_{n\in\omega} g_n^0 = v(a) = f_a^0]\!] = \mathbf{1}.$$

Hence by the Scott lemma g_n converges to f_a a.e. and monotonically.

By the Lebesgue theorem

$$\lim_{n\in\omega} \int g_n \, d\mu = \int f_a \, d\mu = \lambda(\sum_{n\in\omega} a_n).$$

But

$$\lim_{n\in\omega} \int g_n \, d\mu = \lim_{n\in\omega} \sum_{k\leqslant n} \int f_{a_k} \, d\mu = \lim_{n\in\omega} \sum_{k\leqslant n} \lambda(a_k) = \sum_{n<\omega} \lambda(a_n).$$

The rest of the proof is similar and we omit it. ■

COROLLARY. *$\mathscr{R}*\mathscr{R}$ is a measure algebra.*

Moreover, since $\mathscr{R}*\mathscr{R}$ is atomless and σ-generated, $\mathscr{R}*\mathscr{R} \cong \mathscr{R}$.

The above corollary is quoted in [12]. The refinement of the proof of Theorem 1 yields a canonical isomorphism between $\mathscr{R}*\mathscr{R}$ and an algebra formed from the measure space $([0, 1]\times[0, 1], \mathrm{Bor}([0, 1]\times[0, 1]), \lambda\times\lambda)$, where $\lambda\times\lambda$ is a product measure.

2. Boolean factors. We describe some kind of conversion of Theorem 1. Let $\boldsymbol{B}$, $\boldsymbol{A}$ be complete Boolean algebras and $\boldsymbol{B}$ be a complete subalgebra of $\boldsymbol{A}$. Let us describe the method from [9] of constructing the factor $\boldsymbol{A}:\boldsymbol{B}$.

If G is a canonical generic ultrafilter on $\boldsymbol{B}$ in $V^{\boldsymbol{B}}$ then:

$$V^{\boldsymbol{B}} \models \text{“}\underline{G} = \{c \in \check{A}\colon \exists b \in G\ b \leqslant c\} \text{ is a filter on } \check{A}\text{”}.$$

Thus $V^{\boldsymbol{B}} \models$ “$\check{A}/\underline{G}$ is a Boolean algebra”.

In [9] it is proved that $V^{\boldsymbol{B}} \models$ “$\check{A}/\underline{G}$ is c.b.a.”. We call the algebra $\check{A}/\underline{G}$ the *Boolean factor* of $\boldsymbol{B}$ and A and denote this algebra by $A:\boldsymbol{B}$. The term “factor” is used for the reason that $\boldsymbol{B} * (A:\boldsymbol{B}) \cong A$ and whenever $V^{\boldsymbol{B}} \models$ “$\boldsymbol{D}$ is c.b.a.” then $(\boldsymbol{B} * \boldsymbol{D}):\boldsymbol{B} \cong \boldsymbol{D}$ in $V^{\boldsymbol{B}}$.

THEOREM 2. *If $\boldsymbol{B}$ is a complete subalgebra of A and A is a measure algebra then $V^{\boldsymbol{B}} \models$ “$A:\boldsymbol{B}$ is a measure algebra”.*

Proof. Let λ be the measure on A and put $\mu = \lambda \restriction \boldsymbol{B}$. Then μ is a measure on $\boldsymbol{B}$. Now represent $\boldsymbol{B}$ in the form $\mathscr{F}/I_\mu$ for some measure space $(\mathscr{X}, \mathscr{F}, \mu)$. We must find a measure ν in the sense of $V^{\boldsymbol{B}}$, i.e. the Boolean function from $A:\boldsymbol{B}$ into the interval $[0, 1]$ in $V^{\boldsymbol{B}}$. It is easy to see that

$$(*) \qquad [\![d \in (A:\boldsymbol{B})]\!] = \sum_{a \in A} [\![d = [\check{a}]]\!].$$

For $a \in A$ we define a function μ_a on $\boldsymbol{B}$, and hence on $\mathscr{F}$, as follows:

$$\mu_a(b) = \lambda(a \cdot b).$$

Then μ_a is a σ-additive function on $\boldsymbol{B}$ and $\mu_a(b) \leqslant \mu(b)$ for $b \in \boldsymbol{B}$.

Let g_a be the Radon–Nikodym derivative $d\mu_a/d\mu$. Thus $g_a \in M$ and for $B \in \mathscr{F}$ we have:

$$\int_{\boldsymbol{B}} g_a \, d\mu = \mu_a([B]) = \lambda(a \cdot [B]).$$

Now put for $a \in A$

$$\nu([\check{a}]) = g_a^0.$$

The formula $(*)$ ensures that $\nu(d)$ is defined whenever $[\![d \in (A:\boldsymbol{B})]\!] > \mathbf{0}$. We are going to show that $\nu(d)$ is well defined and that ν is really a measure on $A:\boldsymbol{B}$.

Let us recall a useful assertion from [9]: if $b \in \boldsymbol{B}$ and $a, c \in A$ then

$$(**) \qquad b \leqslant [\![[\check{a}] = [\check{c}]]\!] \quad \text{iff} \quad b \cdot a = b \cdot c.$$

(a) ν is well defined: If $b \leqslant [\![[\check{a}] = [\check{c}]]\!]$ then by the formula $(**)$ $b \cdot a = b \cdot c$. Thus $\mu_a(b') = \mu_c(b')$ for $b' \leqslant b$, and so for $[B'] \leqslant [B] = b$

$$\int_{\boldsymbol{B}'} g_a \, d\mu = \int_{\boldsymbol{B}'} g_c \, d\mu.$$

So $g_a = g_c$ almost everywhere in B, i.e.

$$b = [B] \leqslant [\![g_a^0 = g_c^0]\!] = [\![\nu([\check{a}]) = \nu([\check{c}])]\!].$$

(b) v is σ-additive: In [9] it is proved that if $[\![d\in(A:\boldsymbol{B})]\!]=\mathbf{1}$ then there is a unique $c\in A$ such that $[\![d=[\check{c}]]\!]=\mathbf{1}$. To verify σ-additivity it suffices to show that

$$[\![v(\sum_{n\in\check{\omega}} f(n))=\sum_{n\in\check{\omega}} v(f(n))]\!]=\mathbf{1}$$

whenever

$$[\![f\colon \check{\omega}\to(A:\boldsymbol{B}) \text{ and, for } n\neq m,\ f(n)\cdot f(m)=\mathbf{0}]\!]=\mathbf{1}.$$

For $n\in\omega$ let $c_n\in A$ be such that $[\![f(\check{n})=[\check{c}_n]]\!]=\mathbf{1}$. If we put $c=\sum_{n\in\omega} c_n$ then $[\![[\check{c}]=\sum_{n\in\check{\omega}} f(n)]\!]=\mathbf{1}$. So it suffices to show that $[\![g_c^0=\sum_{n\in\check{\omega}} g_{c_n}^0]\!]=\mathbf{1}$. If $b\in\boldsymbol{B}$ then $\sum_{n\in\omega}\mu_{c_n}(b)=\mu_c(b)$. Thus for $[B]=b$ we have $\sum_{n\in\omega}(\int_B g_{c_n}d\mu)=\int_B g_c\,d\mu$. By the Lebesgue theorem

$$\sum_{n\in\omega}(\int_B g_{c_n}d\mu)=\int_B(\sum_{n\in\omega} g_{c_n})d\mu.$$

Thus $\sum_{n\in\omega} g_{c_n}=g_c$ a.e.; hence $[\![g_c^0=\sum_{n\in\check{\omega}} g_{c_n}^0]\!]=\mathbf{1}$. The rest of the proof is similar. ■

We finish this section by the remark that if we apply Theorem 2 to the measure given by Theorem 1, then we obtain the previous measure on $\boldsymbol{D}$ in $V^{\boldsymbol{B}}$.

3. Minimal products. Now we show that the operation of taking the minimal product does not preserve measurability.

For a partially ordered separative set P we denote by $\mathrm{RO}(P)$ the complete Boolean algebra of regular open subsets of P.

Recall that $A\subseteq P$ is *regular open* iff:

(1) A is open, i.e. $(q\in A \text{ and } p\leqslant q)\to p\in A$,

(2) $(\forall q\leqslant p)(\exists r\leqslant q)(r\in A)\to p\in A$.

We write $p\|q$ if $(\exists r\in P)(r\leqslant p \text{ and } r\leqslant q)$ and $p\perp q$ if $\neg(p\|q)$. If $A\in\mathrm{RO}(P)$ then $-A=\{q\in P\colon (\forall p\in A)(p\perp q)\}$. If $A_i\in\mathrm{RO}(P)$ for $i\in I$ then

$$\sum_{i\in I} A_i=\{q\in P\colon (\forall p\leqslant q)(\exists r\leqslant q)(\exists i\in I)(r\in A_i)\}.$$

Define for $p\in P$, $[p]=\{q\in P\colon q\leqslant p\}$. Then the mapping $p\mapsto[p]$ is a dense, order preserving embedding of P into $\mathrm{RO}(P)$.

Let us consider the following partially ordered set: $\mathscr{P}=\{D\in\boldsymbol{R}\colon \lambda(D)>0$, D is a closed set and if G is an open set such that $\lambda(G\cap D)=0$ then $G\cap D=\emptyset\}$ with the ordering: $D_1\leqslant D_2$ iff $D_1\subseteq D_2$.

For $D_1, D_2\in\mathscr{P}$ we have $D_1\|D_2$ iff $\lambda(D_1\cap D_2)>0$.

LEMMA 3.1. *$\mathscr{P}$ is dense in $\mathscr{R}$. Hence $\mathscr{P}$ is separative and $\mathrm{RO}(\mathscr{P})=\mathscr{R}$.*

Proof. We must show that if $b \in \mathscr{R}$ and $b > \mathbf{0}$ then there exists $D \in \mathscr{P}$ with $[D] \leqslant b$. Let $b = [B]$. Let C be a closed subset of B with $\lambda(C) > 0$. Put

$$D = C \setminus \bigcup \{I\colon I \text{ is a rational interval with } \lambda(I \cap C) = 0\}.$$

It is easy to see that $D \in \mathscr{P}$ and $[D] \leqslant b$. ■

In view of the above lemma we can define the minimal product $\mathscr{R}\hat{\otimes}\mathscr{R}$ as $\mathrm{RO}(\mathscr{P} \times \mathscr{P})$, where $\mathscr{P} \times \mathscr{P}$ is the cartesian product of $\mathscr{P}$ ordered coordinatewise. We will write $A \otimes B$ for a pair $\langle A, B\rangle \in \mathscr{P} \times \mathscr{P}$. Let Q be a partially ordered separative set:

$$Q = \{I \subseteq \boldsymbol{R}\colon I \text{ is an open interval with rational endpoints}\}$$

with order: $I_1 \leqslant I_2$ iff $I_1 \subseteq I_2$.

Define $\boldsymbol{C} = \mathrm{RO}(Q)$ (i.e. $\boldsymbol{C}$ is the Cohen algebra).

We will show that $\mathscr{R}\hat{\otimes}\mathscr{R}$ is a product (in the sense of [6]) of two Cohen algebras.

THEOREM 3.2. *There exist two complete embeddings e_1, e_2 such that:*

(1) $e_1, e_2\colon \boldsymbol{C} \to \mathrm{RO}(\mathscr{P} \times \mathscr{P})$,

(2) *if* $a, b \in \boldsymbol{C} \setminus \{\mathbf{0}\}$ *then* $e_1(a) \cdot e_2(b) > \mathbf{0}$,

(3) $e_1'' \boldsymbol{C} \cup e_2'' \boldsymbol{C}$ *completely generates* $\mathrm{RO}(\mathscr{P} \times \mathscr{P})$.

Proof. Let S be a partially ordered set. Call $N \subseteq S$ *maximal* if for every $s \in S$ there is $a \in N$ such that $s \| a$.

LEMMA 3.3. *If $\boldsymbol{B}$ is c.b.a., S is separative, and $h\colon S \to \boldsymbol{B}$ is a function such that:*

(1) *h is 1-1 and order preserving,*

(2) *for $s_1, s_2 \in S$: $s_1 \| s_2$ iff $h(s_1) \cdot h(s_2) > \mathbf{0}$,*

(3) *for every maximal $N \subseteq S$ we have $\sum_{a \in N} h(a) = \mathbf{1}$,*

then there exists a unique embedding $e\colon \mathrm{RO}(S) \to \boldsymbol{B}$ such that for $s \in S$

$$e([s]) = h(s).$$

Proof. Define for $G \in \mathrm{RO}(S)$

$$e(G) = \sum \{h(s)\colon s \in G\}.$$

From the assumptions it follows that e has the required properties. ■

With this in mind define for $I, J \in Q$

$$h_1(I) = \{A \otimes B \in \mathscr{P} \times \mathscr{P}\colon B - A \subseteq \bar{I}\},$$
$$h_2(J) = \{A \otimes B \in \mathscr{P} \times \mathscr{P}\colon B + A \subseteq \bar{J}\},$$

where

$$B - A = \{b - a\colon b \in B, a \in A\}, \quad B + A = \{b + a\colon b \in B, a \in A\},$$

$\bar{I}$ is a closure of I.

(a) If $I, J \in Q$ then $h_1(I), h_2(J) \in \mathrm{RO}(\mathscr{P} \times \mathscr{P})$.

We show this for $h_1(I)$. Of course $h_1(I)$ is open. For regularity assume that $A \otimes B \in \mathscr{P} \times \mathscr{P}$ is such that for every $C \otimes D \leqslant A \otimes B$ there is $E \otimes F \leqslant C \otimes D$ with $F - E \subseteq \bar{I}$. Let $b \in B$, $a \in A$. Then $b - a \in B - A$. Put $I_a^n = [a - 1/n, a + 1/n]$ and similarly for I_b^n. As $A \otimes B \| I_a^n \otimes I_b^n$, for $n \in \omega$, there are sets $E_n \otimes F_n$ with $F_n - E_n \subseteq (I_b^n - I_a^n) \cap \bar{I}$. Hence for $n \in \omega$, $(I_b^n - I_a^n) \cap \bar{I} \neq \emptyset$, but $\bigcap_n (I_b^n - I_a^n) = \{b - a\}$ so $b - a \in \bar{I}$. This shows that $B - A \subseteq \bar{I}$, i.e. $A \otimes B \in h_1(I)$. Thus $h_1(I)$ is regular open as required.

(b) If $N \subseteq Q$ is maximal then $\sum_{I \in N} h(I) = \mathbf{1}$.

Let $A \otimes B \in \mathscr{P} \times \mathscr{P}$. By the theorem of Steinhaus ([10]) there is an open interval J such that $J \subseteq B - A$. As N is maximal there exists $I \in N$ such that $I \cap J \neq \emptyset$. But then we claim that $h_1(I) \cdot [A \otimes B] > \mathbf{0}$.

Pick arbitrary $x_0 \in I \cap J$, x_0 has the form $b - a$ for some $b \in B$ and $a \in A$. Let $n \in \omega$ be such that $I_b^n - I_a^n \subseteq I \cap J$. As $A \otimes B \| I_a^n \otimes I_b^n$, there exists $C \otimes D$ with $C \leqslant A \cap I_a^n$, $D \leqslant B \cap I_b^n$. Thus $C \otimes D \leqslant A \otimes B$ and $D - C \subseteq I_b^n - I_a^n \subseteq I \cap J \subseteq I \subseteq \bar{I}$, i.e. $C \otimes D \in h_1(I)$. This means that $[C \otimes D] \leqslant h_1(I) \cdot [A \otimes B]$.

It is now easy to see that h_1, h_2 satisfy the assumptions of Lemma 3.3 and thus determine complete embeddings e_1, e_2. As $h_1(I) \cdot h_2(J) > \mathbf{0}$ and Q is dense in $\boldsymbol{C}$ we have for $a, b \in \boldsymbol{C} \setminus \{\mathbf{0}\}$

$$e_1(a) \cdot e_2(b) > \mathbf{0}.$$

(c) $h_1'' Q \cup h_2'' Q$ completely generates $\mathrm{RO}(\mathscr{P} \times \mathscr{P})$.

Observe by drawing a picture that for $q \in \boldsymbol{Q}$:

$$(*) \quad \begin{cases} [[q, \infty) \times (-\infty, \infty)] = \sum\limits_{\substack{a,b \in \boldsymbol{Q} \\ b - a = 2q}} h_1((-\infty, a)) \cdot h_2((b, +\infty)), \\ [(-\infty, \infty) \times [q, \infty)] = \sum\limits_{\substack{a,b \in \boldsymbol{Q} \\ b + a = 2q}} h_1((a, +\infty)) \cdot h_2((b, +\infty)). \end{cases}$$

But the left-hand sides of (*) constitute a set of generators for $\mathrm{RO}(\mathscr{P} \times \mathscr{P})$.

The proof of Theorem 3.2 is finished. ■

Remark. The explicit formula for embeddings e_1, e_2 is also

$$e_1(G) = \{A \otimes B\colon B - A \subseteq \bar{G}\}, \quad e_2(G) = \{A \otimes B\colon B + A \subseteq \bar{G}\}.$$

COROLLARY 3.4. *If r_1, r_2 are the canonical random reals in $V^{\mathscr{R} \hat{\otimes} \mathscr{R}}$ and c_1, c_2 are the canonical Cohen reals obtained by e_1, e_2, then*

$$V^{\mathscr{R} \hat{\otimes} \mathscr{R}} \models \text{“}c_1 = r_2 - r_1 \text{ and } c_2 = r_2 + r_1\text{”}.$$

Proof. The equalities (*) in the proof of Theorem 3.2 stated in Boolean-valued terms say that for every $q \in Q$:

$$V^{\mathscr{R}\otimes\mathscr{R}} \models \text{“}\check{q} \leqslant r_1 \text{ iff } 2\check{q} < c_2 - c_1\text{”}$$

and

$$V^{\mathscr{R}\otimes\mathscr{R}} \models \text{“}\check{q} \leqslant r_2 \text{ iff } 2\check{q} < c_2 + c_1\text{”},$$

i.e.

$$V^{\mathscr{R}\hat{\otimes}\mathscr{R}} \models \text{“}c_1 = r_2 - r_1 \text{ and } c_2 = r_2 + r_1\text{”}.$$

COROLLARY 3.5. *C is isomorphic to a complete subalgebra of $\mathscr{R}\hat{\otimes}\mathscr{R}$.*

This corollary was quoted (without proof) in [12].

COROLLARY 3.6. *$\mathscr{R}\otimes\mathscr{R}$ is not a measure algebra.*

Proof. The proof is immediate from Corollary 3.5 and the fact that C is not a measure algebra ([6]).

Now we generalize Corollary 3.5 (and hence Corollary 3.6) to arbitrary atomless measure algebras.

Let us recall a useful lemma from [2].

NEAT COVER LEMMA. *If $(P_1, \leqslant_1)$ and $(P_2, \leqslant_2)$ are two partially ordered sets and φ: $P_1 \to P_2$ is a function such that:*

(1) *φ is "onto",*

(2) *if $p \leqslant_1 q$ then $\varphi(p) \leqslant_2 \varphi(q)$,*

(3) *if $p \leqslant_2 \varphi(q)$ then there is $r \leqslant_1 q$ such that $\varphi(r) = p$,*

then $\mathrm{RO}(P_2)$ is a complete subalgebra of $\mathrm{RO}(P_1)$ (the function φ is called a *neat cover*).

THEOREM 3.7. *If $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ are atomless measure algebras then there exists a complete embedding h: $\boldsymbol{C} \to \boldsymbol{B}_1\hat{\otimes}\boldsymbol{B}_2$. In particular, $\boldsymbol{B}_1\hat{\otimes}\boldsymbol{B}_2$ is not a measure algebra.*

Proof. We can find two embeddings e_1, e_2 such that e_i: $\mathscr{R} \to \boldsymbol{B}_i$ for $i = 1, 2$.

Now we define a neat cover function

$$\psi\colon (\boldsymbol{B}_1 \setminus \{\boldsymbol{0}\}) \times (\boldsymbol{B}_2 \setminus \{\boldsymbol{0}\}) \to (\mathscr{R} \setminus \{0\}) \times (\mathscr{R} \setminus \{0\})$$

by

$$\psi(\langle b_1, b_2\rangle) = \langle \pi_1(b_1), \pi_2(b_2)\rangle$$

where $\pi_i(b) = \prod \{c \in \mathscr{R}\colon e_i(c) \geqslant b\}$.

By the neat cover lemma, $\mathscr{R}\hat{\otimes}\mathscr{R}$ is a complete subalgebra of $\boldsymbol{B}_1\hat{\otimes}\boldsymbol{B}_2$. Now apply Corollary 3.5. ■

4. Other operations. In Section 2 we have observed that two step iteration preserves measurability. So starting from the algebra $\mathscr{R}$, one can construct an increasing sequence of measure algebras:

$$\mathscr{R} < \mathscr{R} * \mathscr{R} < (\mathscr{R} * \mathscr{R}) * \mathscr{R} < \big(((\mathscr{R} * \mathscr{R}) * \mathscr{R}) * \mathscr{R}\big) < \dots$$

Observe that the direct or inverse limit of this sequence is not a measure algebra. The first contains C as a complete subalgebra, the second is not even a c.c.c. algebra.

In Section 3 we dealt with Boolean factors. The usual factors, i.e. by ideals, do not in general preserve measurability.

THEOREM 4.1. *There is an ideal I on $\mathscr{R}$ such that $\mathscr{R}/I \cong C$.*

Proof. First observe that there exists a monomorphism $f\colon C \to \mathscr{R}$. Of course such an f is not complete. The existence of such an f can be deduced from the Sikorski Injectivity Theorem ([6]). So we can think that C is a subalgebra of $\mathscr{R}$. Consequently C is a retract of $\mathscr{R}$, i.e. there is a homomorphism $h\colon \mathscr{R} \to C$ such that h is the identity on C. Now put $I = \{a \in \mathscr{R}\colon h(a) = \mathbf{0}\}$. By the fundamental Boolean algebras theorem $\mathscr{R}/I \cong C$. ■

In Section 3 we developed the minimal product of $\mathscr{R}$. The infinite minimal product of $\mathscr{R}$ is also not a measure algebra, however it is a c.c.c. algebra. We define the strong product $\prod_{n\in\omega} \mathscr{R}$ to be $\mathrm{RO}(P)$, where $P = \{f\colon f\colon \omega \to \mathscr{R}\setminus\{\mathbf{0}\}\}$ with coordinatewise ordering. Similarly we define $\prod_{n\in\omega} C$. We shall show that these two products are isomorphic to a well-known algebra $\mathrm{Coll}(\omega, 2^\omega)$ which collapses 2^ω to ω ([4]).

Now we recall the well-known characterization of the algebra $\mathrm{Coll}(\omega, 2^\omega)$:

THEOREM. *If P is a partially ordered separative set of cardinality 2^ω such that $V^{\mathrm{RO}(P)} \models |(2^\omega)| = \omega$ then $\mathrm{RO}(P) \cong \mathrm{Coll}(\omega, 2^\omega)$.*

COROLLARY 4.2. $\prod_{n\in\omega} C \cong \mathrm{Coll}(\omega, 2^\omega)$.

Remark. The above corollary is quoted in [5]; it was also independently proved by Z. Szczepaniak.

The following proof was found in a manuscript of A. Miller.

Proof. Consider C as $\mathrm{RO}(Q)$ where $Q = (\bigcup_{n\in\omega} {}^n\omega, \supseteq)$. Let Q^ω be the strong product of Q. Let $\langle c_n\colon n \in \omega\rangle$ be the sequence of canonical Cohen reals added by a generic filter G on Q.

Define $z_n \in 2^\omega$ by:

$$z_n(m) = \begin{cases} 1 & \text{if } c_m(c_{m-1}(\dots c_0(n)\dots)) \text{ is even,} \\ 0 & \text{if } c_m(c_{m-1}(\dots c_0(n)\dots)) \text{ is odd.} \end{cases}$$

It is easy to see that

$$V[G] \models 2^\omega \cap V = \{z_n\colon n \in \omega\}. \ \blacksquare$$

From Theorem 3.2 we can deduce

THEOREM 4.3. $\prod_{n\in\omega} \mathscr{R} \cong \mathrm{Coll}(\omega, 2^\omega)$.

Proof. Of course $\left|\prod_{n\in\omega} \mathscr{R}\right| = 2^{\omega}$. Let

$$P = \{f\colon f\colon \omega \to \mathscr{R}\setminus\{\mathbf{0}\}\}.$$

Observe that $P \cong Q$ where

$$Q = \{f\colon f\colon \omega \to (\mathscr{R}\setminus\{\mathbf{0}\})\times(\mathscr{R}\setminus\{\mathbf{0}\})\}.$$

Hence $\prod_{n\in\omega} \mathscr{R} = \prod_{n\in\omega} \mathscr{R}\hat{\otimes}\mathscr{R}$.

By Corollary 3.5 there exists a complete embedding $h\colon C \to \mathscr{R}\hat{\otimes}\mathscr{R}$. Let $S = \{f\colon f\colon \omega \to C\setminus\{\mathbf{0}\}\}$ and define $H\colon Q \to S$ by $H(f)(n) = \pi(f(n))$ where π is a canonical projection induced by h, i.e.

$$\pi(b) = \prod\{c\in C\colon b \leqslant h(c)\}.$$

It is easy to verify that H is a neat cover function, so $\prod_{n\in\omega} C$ is a complete subalgebra of $\prod_{n\in\omega} \mathscr{R}$ and, by completeness, $V^{\prod\mathscr{R}} \models |(2^{\omega})\check{}| = \omega$. ■

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