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A PROOF OF THE TROTTER-KATO THEOREM ON APPROXIMATION OF SEMI-GROUPS

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In this paper a proof of the Trotter-Kato theorem on convergence of semi-groups of linear operators is given, in which this theorem is a corollary of the Hille-Yosida theorem thanks to the treating of the convergent sequence of semi-groups in a space X as one semi-group in a space of convergent sequences of elements of X.

1. Some definitions and theorems from semi-group theory. Let X be a real or complex linear topological space and $\mathscr{L}(X;X)$ the ring of all linear continuous operators of X into X. We say that the operators of the family $\{B_\omega\colon \omega \in \Omega\} \subset \mathscr{L}(X;X)$ are *equicontinuous* iff for every open neighbourhood $\mathscr{U}$ of zero in X there exists an open neighbourhood $\mathscr{V}$ of zero in X such that $B_\omega \mathscr{V} \subset \mathscr{U}$ for every $\omega \in \Omega$. A family $\{S_t\colon t \geqslant 0\} \subset \mathscr{L}(X;X)$ is called a *semi-group* of class (C_0) iff $S_0 = 1$, $S_{t_1}S_{t_2} = S_{t_1+t_2}$ for every $t_1, t_2 \geqslant 0$ and for every $x \in X$ the mapping $t \to S_t x$ of $[0, \infty)$ into X is continuous. The linear operator A with domain $\mathscr{D}(A)$ defined by the conditions

$$\mathscr{D}(A) = \left\{x\colon x \in X,\ \lim_{t\downarrow 0} \frac{1}{t}(S_t x - x) \text{ exists in } X\right\},$$

$$Ax = \lim_{t\downarrow 0} \frac{1}{t}(S_t x - x) \quad \text{for} \quad x \in \mathscr{D}(A),$$

is called the *infinitesimal generator* of the semi-group $\{S_t\colon t \geqslant 0\}$.

If the real or complex linear topological space X is locally convex and sequentially complete and if $\{S_t\colon t \geqslant 0\} \subset \mathscr{L}(X;X)$ is a semi-group of class (C_0) of equicontinuous operators with infinitesimal generator A, then

1° $S_t\mathscr{D}(A) \subset \mathscr{D}(A)$ for every $t \geqslant 0$ and, for every $x_0 \in \mathscr{D}(A)$, the unique continuously differentiable solution of the initial value problem $dx(t)/dt = Ax(t)$ for $t \geqslant 0$, $x(0) = x_0$, is $x(t) = S_t x_0$, and

2° for every $\lambda > 0$ (λ with $\operatorname{Re}\lambda > 0$, if X is complex) the operator $\lambda - A$ has the inverse in $\mathscr{L}(X; X)$ and

$$(\lambda - A)^{-m}x = \frac{1}{(m-1)!}\int_0^\infty e^{-\lambda t} t^{m-1} S_t x dt$$

for every $\lambda > 0$ (λ with $\operatorname{Re}\lambda > 0$), $m = 1, 2, \ldots$, and $x \in X$.

Further, if X is a locally convex, sequentially complete real or complex linear topological space, then, according to the Hille-Yosida theorem, a linear operator A with domain and range in X is the infinitesimal generator of the semi-group $\{S_t : t \geqslant 0\} \subset \mathscr{L}(X; X)$ of class (C_0) of equicontinuous operators iff its domain is dense in X and, for every $\lambda > 0$, the operator $\lambda - A$ has the inverse in $\mathscr{L}(X; X)$ such that the operators $\lambda^m(\lambda - A)^{-m}$ are equicontinuous in $m = 1, 2, \ldots$ and $\lambda > 0$.

2. Deduction of the Trotter-Kato theorem from the Hille-Yosida theorem. Trotter-Kato theorem (see [1] and [2], p. 269) is as follows. Let X be a locally convex, sequentially complete real or complex linear topological space. For every $n = 1, 2, \ldots$, let $\{S_{t,n} : t \geqslant 0\} \subset \mathscr{L}(X; X)$ be a semi-group of class (C_0) with infinitesimal generator A_n such that the operators $S_{t,n}$ are equicontinuous in $t \geqslant 0$ and $n = 1, 2, \ldots$ Suppose that there exists a $\lambda > 0$, if X is a real linear space or a λ with $\operatorname{Re}\lambda > 0$, if X is complex, such that

(i) $\lim_{n\to\infty} (\lambda - A_n)^{-1}x$ exists for every $x \in X$

and

(ii) the set $\{\lim_{n\to\infty}(\lambda - A_n)^{-1}x : x \in X\}$ is dense in X.

Then

(iii) $\lim_{n\to\infty} S_{t,n}x$ exists for every $x \in X$ uniformly in t on every finite interval $[0, T]$.

Proof. Note first that

(iv) the operators $\lambda^m(\lambda - A_n)^{-m}$ are equicontinuous in $\lambda > 0$, $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$

and that (i) holds for every $\lambda > 0$; this follows from our assumptions by 2° (we have

$$\frac{1}{(m-1)!}\int_0^\infty e^{-\lambda t} t^{m-1} dt = \lambda^{-m})$$

and by the identity

$$(\lambda' - A_n)^{-1}x = \sum_{k=0}^{\infty} (\lambda - \lambda')^k (\lambda - A_n)^{-k-1}x$$

for $x \in X$, $n = 1, 2, \ldots$, and $\lambda > 0$, $\lambda' \in (0, 2\lambda)$ if the space X is real, or λ with $\operatorname{Re}\lambda > 0$ and $|\lambda' - \lambda| < \operatorname{Re}\lambda$ if X is complex.

Now, let $\mathscr{X}$ be the linear space of all convergent sequences $\hat{x} = \{x_n\}_{n=1,2,\ldots}$ of elements of X. Let the operators P_n, $n = 1, 2, \ldots$, of $\mathscr{X}$ into X be defined by the identity $\{P_n\hat{x}\}_{n=1,2,\ldots} = \hat{x}$ for every $\hat{x} \in \mathscr{X}$ and let the base of open neighbourhoods of zero in $\mathscr{X}$ consist by definition of all sets

$$\{\hat{x}: \hat{x} \in \mathscr{X},\ P_n\hat{x} \in \mathscr{U} \text{ for every } n = 1, 2, \ldots\},$$

where $\mathscr{U}$ is an arbitrary open neighbourhood of zero in X. Then $\mathscr{X}$ becomes a locally convex sequentially complete linear topological space and condition (iii) may be written in the following equivalent form:

(iii)′ there exists a semi-group $\{\mathscr{S}_t: t \geqslant 0\} \subset \mathscr{L}(\mathscr{X}; \mathscr{X})$ of class (C_0) such that $P_n\mathscr{S}_t = S_{t,n}P_n$ for every $t \geqslant 0$ and $n = 1, 2, \ldots$

We shall show that the operator $\mathscr{A}$ with domain $\mathscr{D}(\mathscr{A})$ defined by the conditions

$$\mathscr{D}(\mathscr{A}) = \{\hat{x}: \hat{x} \in \mathscr{X},\ P_n\hat{x} \in \mathscr{D}(A_n)\} \text{ for every } n = 1, 2, \ldots,$$

$$\{A_nP_n\hat{x}\}_{n=1,2,\ldots} \in \mathscr{X}, \qquad \mathscr{A}\hat{x} = \{A_nP_n\hat{x}\}_{n=1,2,\ldots} \text{ for } \hat{x} \in \mathscr{D}(\mathscr{A}),$$

is the infinitesimal generator of such a semi-group.

Since condition (i) holds for every $\lambda > 0$, we see by (iv) that $\{(\lambda - A_n)^{-1}P_n\hat{x}\}_{n=1,2,\ldots} \in \mathscr{X}$ for every $\lambda > 0$ and $\hat{x} \in \mathscr{X}$, and so, for every $\lambda > 0$, the operator $\lambda - \mathscr{A}$ has the inverse in $\mathscr{L}(\mathscr{X}; \mathscr{X})$ defined by

$$(\lambda - \mathscr{A})^{-1}\hat{x} = \{(\lambda - A_n)^{-1}P_n\hat{x}\}_{n=1,2,\ldots} \quad \text{for every} \quad \hat{x} \in \mathscr{X}.$$

Thus it follows by (iv) that

(v) the operators $\lambda^m(\lambda - \mathscr{A})^{-m}$ are equicontinuous in $\lambda > 0$ and $m = 1, 2, \ldots$

Given $\hat{x} \in \mathscr{X}$ and the open neighbourhood $\mathscr{U}$ of zero in X, there exists, by (ii), an $x \in X$ and a positive integer n_0 such that $P_n\hat{x} - y_n \in \mathscr{U}$ for every $n > n_0$, where $y_n = (\lambda - A_n)^{-1}x$. For every $n = 1, 2, \ldots, n_0$ take a $y_n \in \mathscr{D}(A_n)$ such that $P_n\hat{x} - y_n \in \mathscr{U}$ and put $\hat{y} = \{y_n\}_{n=1,2,\ldots}$. Then $\hat{y} \in \mathscr{D}(\mathscr{A})$ and $P_n(\hat{x} - \hat{y}) \in \mathscr{U}$ for every $n = 1, 2, \ldots$ Hence

(vi) $\mathscr{D}(\mathscr{A})$ is dense in $\mathscr{X}$.

From (v) and (vi), by the Hille-Yosida theorem, we see that $\mathscr{A}$ is the infinitesimal generator of the semi-group $\{\mathscr{S}_t: t \geqslant 0\} \subset \mathscr{L}(\mathscr{X}; \mathscr{X})$ of class (C_0) of equicontinuous operators.

Let $\hat{x} \in \mathscr{D}(\mathscr{A})$. Then, by 1°, $d(\mathscr{S}_t\hat{x})/dt = \mathscr{A}\mathscr{S}_t\hat{x}$ for $t \geqslant 0$, the derivative being taken in the sense of the topology in $\mathscr{X}$ defined above, which implies that

$$\frac{d}{dt}P_n\mathscr{S}_t\hat{x} = P_n\mathscr{A}\mathscr{S}_t\hat{x} = A_nP_n\mathscr{S}_t\hat{x}$$

for $t \geqslant 0$ and $n = 1, 2, \ldots$, the derivative being taken in the topology of X. It follows by 1° that $P_n \mathscr{S}_t \hat{x} = S_{t,n} P_n \hat{x}$ for every $\hat{x} \in \mathscr{D}(\mathscr{A})$, $t \geqslant 0$ and $n = 1, 2, \ldots$, and hence, according to (vi), by continuity, $P_n \mathscr{S}_t = S_{t,n} P_n$ for every $t \geqslant 0$ and $n = 1, 2, \ldots$

REFERENCES

[1] H. F. Trotter, *Approximation of semi-groups of operators*, Pacific Journal of Mathematics 8 (1958), p. 887-919.

[2] K. Yosida, *Functional analysis*, Berlin-Göttingen-Heidelberg 1965.

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