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ON INDUCTIVE LIMITS OF TOPOLOGICAL ALGEBRAS

BY

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Let X be an algebra. A linear space topology σ on X is called *multiplicative* if the map $(x, y) \mapsto xy$ of $X \times X$ into X is (jointly) continuous. X equipped with such a topology is called a *topological algebra*. Clearly, σ is multiplicative iff for every neighbourhood U of zero there exists a neighbourhood V of zero such that $VV \subset U$. If the absolutely convex neighbourhoods U of zero such that $UU \subset U$ form a base of neighbourhoods of zero, then X is called *locally multiplicatively-convex* (*locally m-convex*) [8]. We say that a subset S of a topological algebra is *m-bounded* if for each neighbourhood U of zero there exists a neighbourhood V of zero such that $SV \cup VS \subset U$ or, equivalently, if the maps $x \mapsto xy_0$ and $y \mapsto x_0 y$, where $x_0, y_0 \in S$, are equicontinuous at zero. If S is a bounded subset of X, then S is m-bounded (cf. the proof of Corollary 1). The converse holds when X has a unit.

In his fundamental work [8] on locally m-convex algebras Michael gave some sufficient conditions for the local m-convexity of the algebra X equipped with the linear inductive limit topology associated with an increasing sequence (X_n, σ_n) of locally m-convex subalgebras of X. This study was continued by Warner [13] who gave some other conditions to this effect with many applications.

The present paper deals with a similar problem in the context of the so-called generalized inductive limits of topological algebras.

The notion of generalized inductive limit of locally convex spaces was introduced first by Garling [5] who was inspired by some ideas contained in the earlier work of Wiweger [14]; a careful study of an important particular case was carried out by Roelcke [9]. Extensions of this notion to arbitrary topological linear spaces are due to Turpin (1971) and Adasch and Ernst (1974) (see [10], [11], and [1] for an account of their investigations). In the sequel we shall essentially follow Turpin [11].

Let X be a linear space over the field $\boldsymbol{K}$ of real or complex scalars and let $D = \{a \in \boldsymbol{K}: |a| \leqslant 1\}$. By a *balanced topological space* we mean a balanced subset S (of X) equipped with a topology σ such that the map

$(a, x) \mapsto ax$ of $D \times S$ into S is continuous. By an *inductive system* (of balanced topological spaces) on X we shall understand a sequence

$$\Gamma = (S_n, \sigma_n : n \in \boldsymbol{N})$$

of balanced topological subspaces of X such that

(I_1) $X = \bigcup_{n=1}^{\infty} S_n$;

(I_2) $S_n + S_n \subset S_{n+1}$ and the map $(x, y) \mapsto x+y$ of $S_n \times S_n$ into S_{n+1} is continuous at zero for all $n \in \boldsymbol{N} = \{1, 2, \ldots\}$.

It follows from (I_2) that $S_n \subset S_{n+1}$ and the inclusion map is continuous, i.e., the topology induced by σ_{n+1} on S_n is weaker than σ_n (in symbols: $\sigma_{n+1} | S_n \leqslant \sigma_n$).

Let $\Gamma = (S_n, \sigma_n : n \in \boldsymbol{N})$ be an inductive system on X. We denote by σ_Γ the finest linear topology on X such that $\sigma_\Gamma | S_n \leqslant \sigma_n$ for all $n \in \boldsymbol{N}$. Let $\mathscr{B}_n = \mathscr{B}(\sigma_n)$ be a base of balanced neighbourhoods of zero in (S_n, σ_n). Then the family of all sets

$$U = \sum_{n=1}^{\infty}{}' U_n := \bigcup_{n=1}^{\infty} \sum_{k=1}^{n} U_k,$$

where $U_n \in \mathscr{B}_n$ ($n = 1, 2, \ldots$), is a neighbourhood base of zero for σ_Γ.

Special cases. We shall call Γ

(i) *locally convex* if, for each $n \in \boldsymbol{N}$, S_n is absolutely convex and σ_n has a base of zero consisting of absolutely convex sets (in this case σ_Γ is obviously locally convex);

(ii) *strict* if $\sigma_{n+1} | S_n = \sigma_n$ for all $n \in \boldsymbol{N}$ (in this case $\sigma_\Gamma | S_n = \sigma_n$; [11], p. 41);

(iii) *bounded* if each S_n is a bounded subset of S_{n+1} (i.e., S_n is absorbed by every neighbourhood of zero in S_{n+1});

(iv) *simple* if X is equipped with a linear topology σ and $\sigma_n = \sigma | S_n$ for all $n \in \boldsymbol{N}$;

(v) *bornivorous* if Γ is simple and every σ-bounded subset of X is contained in some S_m;

(vi) *usual* if each S_n is a linear subspace of X and (S_n, σ_n) is a topological linear space.

If Γ_1 and Γ_2 are two inductive systems on X, then we call them *equivalent* and write $\Gamma_1 \sim \Gamma_2$ if $\sigma_{\Gamma_1} = \sigma_{\Gamma_2}$. It is easily seen that if $\Gamma = (S_k, \sigma_k : k \in \boldsymbol{N})$ and $(k_n : n \in \boldsymbol{N})$, $(m_n : n \in \boldsymbol{N})$ are strictly increasing sequences in $\boldsymbol{N}$ such that $k_n \leqslant m_n$ ($n \in \boldsymbol{N}$), then $\Gamma' = (S_{k_n}, \sigma_{m_n} | S_{k_n} : n \in \boldsymbol{N})$ is an inductive system on X and $\Gamma \sim \Gamma'$.

Now we suppose X is a (linear) algebra and let $\Gamma = (S_n, \sigma_n : n \in \boldsymbol{N})$ be an inductive system on X. It is easily seen that if, for every $n \in \boldsymbol{N}$,

$S_n S_n \subset S_{n+1}$ and the map $(x, y) \mapsto xy$ of $S_n \times S_n$ into S_{n+1} is separately continuous, then the multiplication on X is also separately continuous under σ_Γ. As we would like (X, σ_Γ) to be a topological algebra, it is natural to impose somewhat stronger conditions on Γ. We shall therefore say that the system Γ is *algebraic* if

(I_3) $S_n S_n \subset S_{n+1}$ and the map $(x, y) \mapsto xy$ from $S_n \times S_n$ into S_{n+1} is continuous at zero for all $n \in N$.

The system Γ is said to be *m-bounded* if

(m) for every $n \in N$ and every $U \in \mathscr{B}_{n+1}$ there exists $V \in \mathscr{B}_n$ such that $VS_n \cup S_n V \subset U$.

Note that if $S_n S_n \subset S_{n+1}$ for all $n \in N$, then (m) implies (I_3).

Our main result is Theorem 1 which shows that (m) suffices for the multiplicativity of σ_Γ; for a simple and bounded inductive system Γ it is also necessary (Corollary 1). In corollaries to Theorem 1 we indicate also a number of cases where the initial algebraic system Γ does not satisfy (m) but for which an equivalent m-bounded system Γ' can be found.

THEOREM 1. *If $\Gamma = (S_n, \sigma_n : n \in N)$ is an m-bounded inductive system on the algebra X, then (X, σ_Γ) is a topological algebra.*

Proof. Let $p: N \times N \to N$ be an injective map such that $p(1, 1) = 2$ and $p(i, j) \geqslant i+j$ for all $i, j \in N$. Let

$$U = \sum_{n=1}^{\infty}{}' U_n,$$

where $U_n \in \mathscr{B}_n$ for all $n \in N$. We shall find sets $V_n \in \mathscr{B}_n$ such that

$$V_n V_m \subset U_{p(n,m)} \quad \text{for all } n, m \in N. \tag{1}$$

Hence it will follow that the σ_Γ-neighbourhood of zero

$$V = \sum_{n=1}^{\infty}{}' V_n$$

satisfies $VV \subset U$.

For $n = m = 1$ choose $V_1 \in \mathscr{B}_1$ such that $S_1 V_1 \cup V_1 S_1 \subset U_2 = U_{p(1,1)}$; then (1) is satisfied for $n = m = 1$. Suppose we have already found sets $V_i \in \mathscr{B}_i$, $i = 1, 2, \ldots, n$, for some $n \geqslant 1$, so that (1) is satisfied for $1 \leqslant m \leqslant n$. If $1 \leqslant k \leqslant n+1$, then $p(n+1, k) \geqslant n+2$, and hence by (m) we may find $W^{(k)} \in \mathscr{B}_{n+1}$ such that

$$W^{(k)} S_{n+1} \subset U_{p(n+1,k)} \quad \text{and} \quad S_{n+1} W^{(k)} \subset U_{p(k,n+1)}.$$

Let $V_{n+1} \in \mathscr{B}_{n+1}$ be such that

$$V_{n+1} \subset W^{(1)} \cap W^{(2)} \cap \ldots \cap W^{(n+1)}.$$

Now, if $1 \leqslant m \leqslant n+1$, then

$$V_m V_{n+1} \subset S_{n+1} W^{(m)} \subset U_{p(m,n+1)}, \qquad V_{n+1} V_m \subset W^{(m)} S_{n+1} \subset U_{p(n+1,m)}.$$

This completes the proof.

COROLLARY 1. *If Γ is a bounded algebraic inductive system on the algebra X, then (X, σ_Γ) is a topological algebra.*

Proof. If $U \in \mathscr{B}_{n+2}$, then by (I_3) there exists $V \in \mathscr{B}_{n+1}$ such that $VV \subset U$. Since S_n is bounded in S_{n+1}, there exists $a \in (0, 1)$ such that $aS_n \subset V$. Choose $W \in \mathscr{B}_n$ so that $W \subset aV$. Then

$$S_n W \subset S_n(aV) = (aS_n)V \subset VV \subset U$$

and, similarly, $WS_n \subset U$. It follows that the inductive system

$$\Gamma_1 = (S_{2n-1}, \sigma_{2n-1} : n \in N)$$

is m-bounded. Evidently, $\Gamma \sim \Gamma_1$, and so we may apply Theorem 1, which completes the proof.

COROLLARY 2. *Let Γ be a bounded simple inductive system on the algebra X. Then σ_Γ is multiplicative iff Γ is m-bounded.*

COROLLARY 3. *Let (X, σ) be a topological algebra with a fundamental sequence of bounded sets and let τ^* be another multiplicative topology on X such that $\tau^* \leqslant \sigma$. Then the finest linear topology γ on X agreeing with τ^* on all σ-bounded sets is multiplicative.*

Proof. From the assumption on (X, σ) it follows that it has a fundamental sequence $(S_n : n \in N)$ of bounded balanced sets such that $(S_n + S_n) \cup \cup (S_n S_n) \subset S_{n+1}$ for all $n \in N$. Then $\gamma = \tau_\Gamma^*$, where $\Gamma = (S_n, \tau^* | S_n : n \in N)$, and so it is enough to apply Corollary 1.

Example. Let $C(S)$ be the topological algebra of all bounded and continuous (real- or complex-valued) functions on a locally compact Hausdorff space S equipped with the sup-norm topology σ. From Corollary 3 it follows immediately that the strict topology β on $C(S)$ (cf. [3]), i.e., the finest locally convex topology on $C(S)$ agreeing with the compact-open topology on all σ-bounded sets, is multiplicative. For another proof see [3], p. 152.

Let (Y, ϑ) be a topological linear space. Then $\mathscr{B}(\vartheta)$ will denote a (fixed) base of balanced ϑ-neighbourhoods of zero and $\mathrm{Bd}(\vartheta)$ the class of all ϑ-bounded subsets of Y.

THEOREM 2. *Let $\Gamma = (S_n, \sigma_n : n \in N)$ be a usual inductive system of topological algebras on X such that*

$$\mathscr{B}(\sigma_n) \cap \mathrm{Bd}(\sigma_{n+1}) \neq \emptyset \quad \textit{for each } n \in N. \tag{2}$$

Then (X, σ_Γ) is a topological algebra.

Proof. Let $U_n \in \mathscr{B}(\sigma_n) \cap \mathrm{Bd}(\sigma_{n+1})$ for each $n \in \boldsymbol{N}$. First we shall construct an inductive system $\Gamma_1 = (A_n, \sigma_n | A_n \colon n \in \boldsymbol{N})$ such that $\Gamma \sim \Gamma_1$, where $A_n \in \mathscr{B}(\sigma_n) \cap \mathrm{Bd}(\sigma_{n+1})$ and also $(A_n + A_n) \cup (A_n A_n) \subset A_{n+1}$ for all $n \in \boldsymbol{N}$.

Set $A_1 = U_1$. Suppose we have already defined $A_1, A_2, \ldots, A_n$ in such a way that the desired conditions are satisfied. Since $(A_n + A_n) \cup \cup (A_n A_n)$ is σ_{n+1}-bounded, it is contained in $a U_{n+1}$ for some $a > 0$. Set $A_{n+1} = a U_{n+1}$. It is obvious that Γ_1 is an algebraic and bounded inductive system on X. We have also $\Gamma \sim \Gamma_1$, as is seen from the following simple fact (cf. [5]): If α, β are two linear topologies on a linear space and U is a β-neighbourhood of zero, then $\alpha \leqslant \beta$ iff $\alpha | U \leqslant \beta | U$. Finally, by Corollary 1, the topology σ_Γ is multiplicative.

Remark. Condition (2) is clearly satisfied when each (S_n, σ_n) is locally bounded or when the inclusion map of S_n into S_{n+1} is compact (or precompact) for each $n \in \boldsymbol{N}$.

COROLLARY 1. *The (linear topological) direct sum of a sequence of locally bounded topological algebras is a topological algebra.*

Remark. The (linear topological) direct sum of a sequence of locally m-convex algebras is a locally m-convex algebra (cf. Example 9 of [13]).

A topological linear space (X, σ) is called an *Ultra-L-space* (respectively, *Ultra-Lb-space*) if $\sigma = \sigma_\Gamma$ for every simple (respectively, bornivorous) inductive system Γ on X. Every ultrabarrelled space is an Ultra-L-space and every quasi-ultrabarrelled space is an Ultra-Lb-space. It is easily seen that every simple inductive system on an Ultra-L-space is bornivorous. For the basic properties of spaces of this type we refer to [1], [6], and [7].

THEOREM 3. *Let Γ be a usual inductive system on the algebra X consisting of topological algebras (S_n, σ_n) each of which is an Ultra-Lb-space with a fundamental sequence of bounded sets. Then (X, σ_Γ) is a topological algebra.*

Proof. Let $(B_m^{(n)} \colon m \in \boldsymbol{N})$ be an increasing fundamental sequence of σ_n-bounded balanced sets in S_n. Let $A_1 = B_1^{(1)}$. Suppose for some $n \in \boldsymbol{N}$ we have already chosen sets $B_i \in \mathrm{Bd}(\sigma_i)$ so that $(A_i + A_i) \cup (A_i A_i) \subset A_{i+1}$ for $i = 1, 2, \ldots, n$. Since A_n is σ_{n+1}-bounded, there exists $p \in \boldsymbol{N}$ such that $(A_n + A_n) \cup (A_n A_n) \subset B_p^{(n+1)}$. Then define

$$A_{n+1} = B_{n+1}^{(1)} + B_{n+1}^{(2)} + \ldots + B_{n+1}^{(n+1)} + B_p^{(n+1)}.$$

Let $\Gamma_1 = (A_n, \sigma_n | A_n \colon n \in \boldsymbol{N})$. It is obvious that $\sigma_\Gamma \leqslant \sigma_{\Gamma_1}$. Now fix $k \in \boldsymbol{N}$. Then

$$B_n^{(k)} \subset A_n \quad \text{and} \quad \sigma_{\Gamma_1} | B_n^{(k)} \leqslant \sigma_k | B_n^{(k)}$$

for all $n \geqslant k$. Since (S_k, σ_k) is an Ultra-Lb-space, $\sigma_{\Gamma_1} | S_k \leqslant \sigma_k$ for all $k \in \boldsymbol{N}$.

Hence $\sigma_{\Gamma_1} \leqslant \sigma_\Gamma$. Thus $\sigma_{\Gamma_1} = \sigma_\Gamma$ and it suffices to apply Corollary 1 to Theorem 1.

Now let X be an algebra with a locally convex topology σ. Such an algebra (X, σ) is called *inverse continuous* if it has a unit e, the multiplicative group $G(X)$ of invertible elements is open, and the map $x \mapsto x^{-1}$ is continuous on $G(X)$. It is known (and easy to see) that if the map $x \mapsto x^{-1}$ is continuous at e, then it is continuous on $G(X)$. By a theorem due to Turpin (cf. [12], p. 123), every commutative inverse continuous locally convex topological algebra is locally m-convex.

We shall need the following lemma proved in [2].

LEMMA. *Let X be an algebra with the unit e and let $\Gamma = (S_n, \sigma_n : n \in N)$ be a locally convex m-bounded inductive system on X such that for each $n \in N$*

(a) $S_n S_n \subset S_{n+1}$,

(b) *S_n is contained in a subalgebra X_n of X,*

(c) $X_1 \subset X_2 \subset \ldots,\ X = \bigcup_{n=1}^{\infty} X_n$,

(d) *$\sigma_n = \tau_n | S_n$, where τ_n is a locally convex topology on X_n.*

Assume that

(*) *for every $n \in N$ there exist $V \in \mathscr{B}(\tau_n)$ and $m \in N$ such that*

$$e + V \cap S_n \subset G(X) \quad \text{and} \quad (e + V \cap S_n)^{-1} \subset S_m.$$

Then (X, σ_Γ) is inverse continuous.

THEOREM 4. *Let X be an algebra with the unit e and assume that $\Gamma = (S_n, \sigma_n : n \in N)$ is the usual inductive system of inverse continuous topological algebras (S_n, σ_n) on X. Suppose also that*

$$\mathscr{B}(\sigma_n) \cap \mathrm{Bd}(\sigma_{n+1}) \neq \emptyset \quad \text{for all } n \in N.$$

Then (X, σ_Γ) is an inverse continuous topological algebra.

Proof. Without loss of generality we may assume that $e \in S_n$ for all $n \in N$. Hence, by assumption, for each $n \in N$ and each $U \in \mathscr{B}(\sigma_n)$ there exists $V \in \mathscr{B}(\sigma_n)$ such that

$$e + V \subset G(S_n) \subset G(X) \quad \text{and} \quad (e+V)^{-1} \subset e + U.$$

As in the proof of Theorem 2 we can construct an algebraic bounded inductive system $\Gamma_1 = (A_n, \sigma_n | A_n : n \in N)$ such that $\Gamma \sim \Gamma_1$, where

$$A_n \in \mathscr{B}(\sigma_n) \cap \mathrm{Bd}(\sigma_{n+1}) \quad \text{for all } n \in N.$$

On the other hand, $\Gamma_2 = (A_{2n-1}, \sigma_{2n-1} | A_{2n-1} : n \in N)$ is an m-bounded inductive system and $\Gamma_2 \sim \Gamma_1$ (see the proof of Corollary 1). Hence it is enough to prove that Γ_1 satisfies condition (*) from the Lemma. Fix $n \in N$. Since $A_n \in \mathscr{B}(\sigma_n)$ and (S_n, σ_n) is inverse continuous, there exists

$V \in \mathcal{B}(\sigma_n)$ such that

$$e+V \subset G(X) \quad \text{and} \quad (e+V)^{-1} \subset e+A_n.$$

Hence

$$e+V \cap A_n \subset G(X) \quad \text{and} \quad (e+V\cap A_n)^{-1} \subset A_m$$

for some $m \in N$.

Applying the above-mentioned result of Turpin we now prove the following

COROLLARY 1. *If Γ is the usual inductive system of commutative Banach algebras on the algebra X with a unit, then (X, σ_Γ) is a locally m-convex algebra.*

Remark. In general, the usual inductive limit of locally m-convex algebras need not be locally m-convex (cf. Example 6 of [13]).

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