COLLOQUIUM MATHEMATICUM

XVIII DÉDIÉ À M. TADEUSZ WAŻEWSKI 1967

ON PROJECTIVE SPACES AND RESOLUTIONS IN CATEGORIES OF COMPLETELY REGULAR SPACES

BY

J. MIODUSZEWSKI AND L. RUDOLF (WROCŁAW)

We shall give here a reduction of the theory of projective spaces and resolutions in categories consisting of completely regular spaces and continuous mappings to that of their full subcategories consisting of all their compact spaces. The last one, chronologically earlier than the first, due to Gleason [4] and Rainwater [9], is quite formal. The first one was considered by Iliadis [7], Ponomarev [8] and Flachsmeyer [1]. The reduction consists only in categorial considerations. Let us remark, however, that some of the results of the last three papers are valid for Hausdorff or regular spaces, too.

1. Preliminaries from the theory of categories. Let $\mathscr{C}$ be a category and $\mathscr{A}$ a class of morphisms of $\mathscr{C}$. An object P is said to be *$\mathscr{A}$-projective* if for each morphism $Y \to X$ from $\mathscr{A}$ and each morphism $P \to X$ there exists a morphism filling up the diagram (¹)

$$\begin{tikzcd} P \arrow[d] \arrow[dr, dashed] & \\ X & Y \arrow[l] \end{tikzcd}$$

A morphism $Z \to Y$ is said to be an *$\mathscr{A}$-injection* (²) if for each of its decompositions $Z \to Z' \to Y$ the morphism $Z \to Z'$ is an isomorphism whenever it belongs to $\mathscr{A}$. Clearly, if an $\mathscr{A}$-injection belongs to $\mathscr{A}$, it is an isomorphism.

A morphism $Y \to X$ from $\mathscr{A}$ is said to be *$\mathscr{A}$-irreducible* if each $\mathscr{A}$-injection $Z \to Y$ is an isomorphism whenever $Z \to Y \to X$ belongs to $\mathscr{A}$.

A morphism $P \to X$ from $\mathscr{A}$ is said to be an *$\mathscr{A}$-projective resolution* of X if it is $\mathscr{A}$-irreducible and P is $\mathscr{A}$-projective.

(¹) to be commutative, of course.

(²) For this notion and related questions see J. Michalski (a paper in preparation).

Consider two assumptions on $\mathscr{A}$, namely that

(1) *a selection* (³) *is an isomorphism whenever it belongs to* $\mathscr{A}$,

(2) *a retraction is an isomorphism whenever it is an* $\mathscr{A}$*-injection.*

We deduce from (1) that

(3) *A selection is always an* $\mathscr{A}$*-injection.*

In fact, if $Z \to Y$ is a selection and $Z \to Z' \to Y$ is its decomposition, where $Z \to Z'$ belongs to $\mathscr{A}$, then $Z \to Z'$, being a selection, is, by (1), an isomorphism.

Although assumptions (1) and (2) seem to be casual, yet they imply a number of useful properties of $\mathscr{A}$-projective resolution, e. g. a kind of uniqueness.

Call a morphism f a (*strong*) *semi-monomorphism* if the cancellation law

$$f \circ g = f \Rightarrow g \text{ is (the identity) an isomorphism}$$

holds.

1.1. *If* $p\colon P \to X$ *is a semi-monomorphism belonging to* $\mathscr{A}$ *and* P *is* $\mathscr{A}$*-projective, then* p *is an* $\mathscr{A}$*-projective resolution of* X.

It suffices to prove that p is $\mathscr{A}$-irreducible. To do this let $Z \to P$ be an $\mathscr{A}$-injection such that $Z \to P \xrightarrow{p} X$ belongs to $\mathscr{A}$. Since P is $\mathscr{A}$-projective, we may fill up the diagram

$$\begin{tikzcd} P \arrow[d, "p"'] \arrow[drr] & & \\ X & P \arrow[l, "p"] & Z \arrow[l] \end{tikzcd}$$

Since p is a semi-monomorphism, $P \to Z \to P$ is an isomorphism. Hence $Z \to P$ is a retraction. By (2), $Z \to P$ is an isomorphism.

1.2. *If* $p\colon P \to X$ *is a semi-monomorphism from* $\mathscr{A}$ *and* P *is* $\mathscr{A}$*-projective, then* $p\colon P \to X$ *is isomorphic with any* $\mathscr{A}$*-projective resolution* $q\colon Q \to X$ *of* X.

To prove this, take morphisms $P \to Q$ and $Q \to P$ such that $P \xrightarrow{p} X = P \to Q \xrightarrow{q} X$ and $Q \xrightarrow{q} X = Q \to P \xrightarrow{p} X$, the existence of which follows from the fact that P and Q are $\mathscr{A}$-projective and that p and q belong to $\mathscr{A}$. We easily get $P \xrightarrow{p} X = P \to Q \to P \xrightarrow{p} X$. Since p is a semi-monomorphism, $P \to Q \to P$ is an isomorphism. Thus $P \to Q$ is a selection. By (3), $P \to Q$ is an $\mathscr{A}$-injection. Since $P \to Q \xrightarrow{q} X$ belongs to $\mathscr{A}$ and q is $\mathscr{A}$-irreducible, $P \to Q$ is an isomorphism.

(³) Selection, a dual notion to the retraction, is a morphism having a left inverse.

In the case when all the $\mathscr{A}$-projective resolutions of X are mutually isomorphic, we denote each of them by a^X: $a^X \to X$ (we omit here the symbol of the class $\mathscr{A}$).

Clearly, for each $Y \to X$, where Y is $\mathscr{A}$-projective, there exists a morphism filling up the diagram

$$\begin{tikzcd} Y \arrow[d] \arrow[dr, dashed] & \\ X & aX \arrow[l, "a^X"] \end{tikzcd} \tag{4}$$

If the category $\mathscr{C}$ and the class $\mathscr{A}$ are such that $Y \to aX$ is uniquely determined by $Y \to X$ and if the composition $aY \to Y \to X$ belongs to $\mathscr{A}$, the operation a (defined up to now only on objects) is of a functorial character. Namely, to a morphism $Y \to X$ there corresponds a morphism $aY \to aX$ uniquely determined by $aY \to Y \to X$ in the diagram of the form (4), so that a is defined on morphisms. Now, it is easy to check that a is a functor.

Another notion of the theory of categories we shall use is that of *uniformization* of a pair of morphisms (4) f, g: $Y, Z \rightrightarrows X$ of the category: it consists of a pair of morphisms h, k: $T \rightrightarrows Y, Z$ such that the diagram

$$\begin{tikzcd} T \arrow[r, dashed, "h"] \arrow[d, "k"'] & Y \arrow[d, "f"] \\ Z \arrow[r, dashed, "g"'] & X \end{tikzcd}$$

is commutative and such that for each pair h', k': $T' \rightrightarrows Y, Z$ having the same property there exists a *unique* morphism u: $T' \to T$ such that $h \circ u = h'$ and $k \circ u = k'$. These conditions determine h and k uniquely up to an isomorphism. Cf. e. g. pullback diagram in [2], p. 40; a uniformization of two morphisms may be regarded as a special case of a more general notion of the greatest lower bound or inverse limit of a system of objects and morphisms of the category.

Call h an *inverse image* of g under f, in symbols, $h = f^{-1}(g)$.

Let us note the following cancellation law for uniformizations, which is quite easy to prove:

1.3. *If* u, v: $W \rightrightarrows T$ *are such that* $h \circ u = k \circ v$, *then* $u = v$.

(4) The symbol $A, B \underset{v}{\overset{u}{\rightrightarrows}} C, D$ denotes two morphisms u: $A \to C$ and v: $B \to D$. We shall write also u, v: $A, B \rightrightarrows C, D$. If $A = B$, the symbol will be written shortly u, v: $A \rightrightarrows C, D$. An analogous abbreviation will be used in the case of $C = D$. So, the meaning of the symbols $A, B \rightrightarrows C, D \rightrightarrows E, F$, which will appear later, is clear.

2. Some facts and informations concerning the compact case. Let $\mathscr{CCR}$ be a category ([5]) consisting of compact completely regular spaces, i.e. compact Hausdorff spaces, and continuous mappings.

By a *projective* (*compact c.r.*) *space* we mean an $\mathscr{A}$-projective object of $\mathscr{CCR}$, where $\mathscr{A}$ is the class of all mappings ([6]) *onto* $\mathscr{CCR}$.

The following characterization of projective spaces is rather formal. In the proof we shall omit the trivial implication.

2.1. *Projective* (*compact c.r.*) *spaces are the same as spaces each mapping onto which is a retraction.*

Let Q be a space each mapping onto which is a retraction. To prove that Q is projective, let $f\colon Y \xrightarrow{\text{onto}} X$ and $g\colon Q \to X$ be given. Take a uniformization

$$\begin{array}{ccc} Z & \dashrightarrow & Q \\ \downarrow & & \downarrow g \\ Y & \xrightarrow[f]{} & X \end{array}$$

of f and g. Since $f\colon Y \to X$ is onto, $Z \to Q$ is onto, too (an easy property of the inverse image, true in each "sufficiently complete" category of sets). Hence, by assumption, $Z \to Q$ is a retraction. Taking an arbitrary right inverse of it, $Q \to Z$, we get a mapping $Q \to Y = Q \to Z \to Y$ such that $Q \to Y \xrightarrow{f} X = Q \xrightarrow{g} X$. This proves that Q is projective.

Remark. The space Z is in fact the subset $\{(y, q)\colon f(y) = g(q)\}$ of the product $Y \times Q$, and it is compact whenever Y and Q are compact and f and g are continuous. The category $\mathscr{CCR}$ considered in theorem 2.1 ought have the property: if f and g belong to $\mathscr{CCR}$, then the uniformization of f and g belongs to $\mathscr{CCR}$, too. In the case of compact c.r. spaces this assumption is of a purely formal character. In the proofs which follow, we shall assume that categories are such that the operations pointed in the proofs may be performed. Of course, we shall assume this only in cases when such assumptions are of purely formal character, e.g. as in the just described situation.

The following characterization will be used in the sequel only exceptionally:

2.2 (Gleason [4]). *Projective* (*compact c. r.*) *spaces are the same as extremally disconnected spaces.*

Recall that a space is *extremally disconnected* if the closure of each open subset is open.

([5]) We shall not consider the category of all compact c.r. spaces (c.r. for completely regular) and their all continuous mappings. Although such a category is not antinomial, yet there is no need to use it in practice.

([6]) continuous, of course.

The class $\mathscr{A}$ of all mappings onto belonging to $\mathscr{CCR}$ satisfies conditions (1) and (2) of § 1. It is obvious that condition (1) holds. To prove (2) it suffices to know that

(5) *$\mathscr{A}$-injections are one-to-one.*

To prove this, let $h\colon Z \to Y$ be an $\mathscr{A}$-injection. Suppose, on the contrary, that for some distinct points a and b we have $h(a) = h(b)$. Let R be an equivalence relation on Z the equivalence classes of which are $\{a, b\}$ and single points. Take the decomposition $Z \to Z/R \to Y$ of h (which exists in a "sufficiently complete" $\mathscr{CCR}$). Since h is an $\mathscr{A}$-injection, $Z \to Z/R$ is an isomorphism. We have a contradiction.

From (5) it follows, in virtue of compactness, that $\mathscr{A}$-injections are embeddings, even closed embeddings. This explains the meaning of $\mathscr{A}$-irreducible morphisms of $\mathscr{CCR}$ which will be called *irreducible mappings*. By a *projective resolution* of a space X of $\mathscr{CCR}$ we mean an irreducible mapping $P \xrightarrow{\text{onto}} X$, where P is a projective space.

The existence and uniqueness up to an isomorphism of projective resolutions for each (compact c.r.) space was shown in [4] and [9]. More precisely,

2.3 (Rainwater [9]). *Each compact c.r. space admits a projective resolution which is a strong semi-monomorphism.*

We know that this implies the uniqueness up to an isomorphism of projective resolutions, according to 1.2 and the fact that $\mathscr{A}$ satisfies conditions (1) and (2) of § 1.

3. Projective completely regular spaces. Let $\mathscr{CR}$ be a category consisting of completely regular spaces and continuous mappings. Assume that $\mathscr{CR}$, as well as its full ([7]) subcategory including all compact spaces, call this subcategory $\mathscr{CCR}$, satisfy some „completeness" conditions as in § 2.

Let $\mathscr{A}$ be the class of all *perfect* mappings of $\mathscr{CR}$, i. e. mappings $f\colon Y \xrightarrow{\text{onto}} X$ such that the induced mappings $\beta f\colon \beta Y \to \beta X$ in the Čech-Stone compactification transform $\beta Y - \beta_Y(Y)$ into (in fact, onto) $\beta X - \beta_X(X)$ (if Z is a space, we denote by $\beta_Z\colon Z \to \beta Z$ its Čech-Stone embedding). Perfect mappings were characterized by Henriksen and Isbell [5] to be closed mapping each inverse image $f^{-1}(x)$, $x \in X$, of which is compact.

By a *projective* (*completely regular*) *space* we mean an $\mathscr{A}$-projective object of $\mathscr{CR}$, where $\mathscr{A}$ is the class just defined.

Note that if $\mathscr{A}$ is the class of all mappings onto of $\mathscr{CR}$ (not necessarily perfect), then $\mathscr{A}$-projective objects are simply discrete spaces. By restricting the considerations to the class of perfect mappings we exclude this trivial solution.

([7]) A subcategory is said to be *full* if it contains morphisms of the category whenever it contains initial and final objects of them.

Before we pass to the characterization of projective c.r. spaces, we shall show a property of inverse images.

(6) *If $g\colon Y \to X$ is an embedding and $f\colon Z \to X$ is a mapping, then $f^{-1}(g)$ is an embedding.*

To prove this, consider the diagram where $h = f^{-1}(g)$. In fact, since g is one-to-one, $f^{-1}(g)$ is one-to-one, too. Consider the decomposition $T \to Y' \to Y$ of $f^{-1}(g)$, where $Y' \to Y$ is an embedding and, in consequence, $T \to Y'$ is one-to-one and onto. There exists a mapping (in the

$$\begin{tikzcd}
& Y' \arrow[dr] \arrow[ddl, dashed, "h"] & \\
T \arrow[ur] \arrow[rr] \arrow[d, "k"'] & & Y \arrow[d, "f"] \\
Z \arrow[rr, "g"'] & & X
\end{tikzcd}$$

set-theoretical sense) $u\colon Y' \to Z$ such that $T \to Y' \xrightarrow{u} Z = T \xrightarrow{k} Z$. The mapping $u\colon Y' \to Z$ is continuous, for if V is an open subset of Z and V' an open subset of X such that $g^{-1}(V') = V$, then $u^{-1}(V) = Y' \cap g^{-1}(V')$, $g^{-1}(V')$ being an open subset of Y'.

Mappings $Y' \rightrightarrows Y, Z$ induce a mapping $Y' \to T$ commuting with them in the uniformization diagram. We easily get $T \xrightarrow{h} Y = T \to Y' \to \to T \xrightarrow{h} Y$ and $T \xrightarrow{k} Z = T \to Y' \to T \xrightarrow{k} Z$. By uniqueness assumed in the definition of uniformization, $T \to Y' \to T$ is an identity. Hence $T \to Y'$ is a selection. Being onto, $T \to Y'$ is an isomorphism. Thus $T \to Y$ is an embedding.

3.1. *Projective c. r. spaces are the same as spaces each perfect mapping onto which is a retraction.*

We prove only the non-trivial implication.

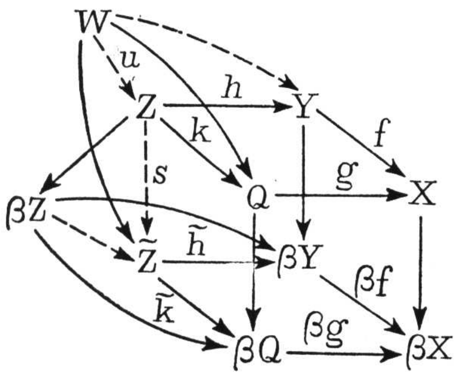
Let Q be a space each perfect mapping onto which is a retraction. To prove that Q is projective, let a perfect mapping $f\colon Y \to X$ and a mapping $g\colon Q \to X$ be given. Take a uniformization of f and g and a uniformization of βf and βg, being the "top" and the "bottom" face of the "cubic" diagram given on p. 191 (further arrows will be explained later).

Mapping $Z \to \tilde{Z}$ is induced by $Z \to Q \to \beta Q$ and $Z \to Y \to \beta Y$.

We show first that

(7) *The "left vertical" face $Z \rightrightarrows \tilde{Z}$, $Q \rightrightarrows \beta Q$ of the "cubic" diagram is a uniformization of the pair $\tilde{Z}$, $Q \rightrightarrows \beta Q$.*

To do this, let $W \rightrightarrows \tilde{Z}$, Q be such that $W \to \tilde{Z} \to \beta Q = W \to Q \to \beta Q$. Consider $W \to \tilde{Z} \to \beta Y$. The image of W under this mapping is contained in $\beta_Y(Y)$ for in the other case the mapping $\beta Y \xrightarrow{\beta f} \beta X$ (which transforms $\beta Y - \beta_Y(Y)$ into $\beta X - \beta_X(X)$, f being perfect) would transform some points of the image of W into $\beta X - \beta_X(X)$, but, on the other hand, W is transformed by $W \to Q \to \beta Q \to \beta X$ into $\beta_X(X)$.



Hence, a mapping $W \to Y$ may be defined such that $W \to Y \to X = W \to Q \to X$ (we get this commutativity after a cancellation by the monomorphism $X \to \beta X$).

Now, mappings $W \rightrightarrows Y, Q$ induce a mapping $W \xrightarrow{u} Z$ such that $W \xrightarrow{u} Z \to Q = W \to Q$ and $W \xrightarrow{u} Z \to Y = W \to Y$. To get (7) we must have equations $W \xrightarrow{u} Z \to Q = W \to Q$ and $W \xrightarrow{u} Z \to \tilde{Z} = W \to \tilde{Z}$. It remains to show the second one.

This is a consequence of the equations $W \to \tilde{Z} \xrightarrow{\tilde{k}} \beta Q = W \xrightarrow{u} Z \xrightarrow{s} \tilde{Z} \xrightarrow{\tilde{k}} \beta Q$ and $W \to Z \xrightarrow{\tilde{h}} \beta Y = W \xrightarrow{u} Z \xrightarrow{s} \tilde{Z} \to \beta Y$ which follow from the commutativity properties of the diagram known up to now. These equations allow, in virtue of 1.3, a cancellation by $\tilde{h}$ and $\tilde{k}$, simultaneously, since $\tilde{h}$ and $\tilde{k}$ fill up the uniformization on the "bottom" face of the diagram.

Now, since the "left vertical" face of the diagram is a uniformization and $Q \to \beta Q$ is an embedding, $Z \xrightarrow{s} \tilde{Z}$ is an embedding, by (6).

The next step consists in the proof of the equation

$$s(Z) = \tilde{k}^{-1}\big(\beta_Q(Q)\big), \tag{8}$$

the inclusion $\subset$ of which is obvious.

Suppose, on the contrary, that there exists a point $a \in \tilde{k}^{-1}\big(\beta_Q(Q)\big) - s(Z)$. Let S be a one-point space and let i, j: $S \rightrightarrows \tilde{Z}, Q$ be given by $i(S)$ and $j(S) = \tilde{k}(a)$. We have $S \to \tilde{Z} \to \beta Q = S \to Q \to \beta Q$. But there is no mapping $S \to Z$ such that $S \xrightarrow{i} \tilde{Z} = S \to Z \xrightarrow{s} \tilde{Z}$, because the image of $S \to Z \xrightarrow{s} \tilde{Z}$ is $s(Z)$ and the image of $S \xrightarrow{i} \tilde{Z}$ is $a \notin s(Z)$. We have a contradiction with the properties of uniformizations.

Let $\beta Z \to \tilde{Z}$ be the induced mapping of $Z \xrightarrow{s} \tilde{Z}$ in the Čech-Stone compactification. The composite mapping $\beta Z \to \tilde{Z} \to \beta Q$ is equal to $\beta Z \to \beta Q$, the induced mapping of $Z \to Q$, in virtue of the uniqueness property of the Čech-Stone compactification.

Since $s\colon Z \to \tilde{Z}$ is an embedding, $\overline{h(Z)}$ is a compactification of Z, and hence the mapping $\beta Z \to \tilde{Z}$ transforms $\beta Z - \beta_Z(Z)$ into $\overline{h(Z)} - h(Z)$. By (8), $\overline{h(Z)} - h(Z)$ is transformed into $\beta Q - \beta_Q(Q)$.

This proves that $Z \to Q$ is perfect.

Then, by the assumption on Q, $Z \to Q$ is a retraction. The rest of the proof of 3.1 is a formal consideration, the same as that of 2.1.

Let us exemplify our method by proving another characterization of projective c.r. spaces, namely that they are the same as extremally disconnected spaces (see [1]), reducing the proof to the compact case.

The following lemma may be found in book [3]:

(9) *If E is an e.d. (e.d. for extremally disconnected) space, then βE is an e.d. space.*

3.2. *Each perfect mapping onto an e. d. space is a retraction.*

In fact, let E be an e. d. space and let $f\colon Y \to E$ be a perfect mapping. Pass to $\beta f\colon \beta Y \to \beta E$. By (9), βE is e. d. and hence projective in the subcategory $\mathscr{CCR}$, by 2.2. Then by the trivial implication of 2.1, βf is a retraction. Let $\beta E \to \beta Y$ be one of the right inverses of βf. Since f is perfect, βf transforms $\beta Y - \beta_Y(Y)$ into $\beta E - \beta_E(E)$. This allows to define a mapping $E \to Y$ by cutting $\beta E \to \beta Y$ to $\beta_E(E)$. Mapping $E \to Y$ is a right inverse for $f\colon Y \to E$.

3.3 (Flachsmayer [1]). *Projective c.r. spaces are the same as extremally disconnected spaces.*

It is an immediate consequence of 3.1 and 3.2 that e.d. spaces are projective. The proof of the converse implication is an easy consideration, the same as that of the analogous implication in the compact case (see [4]).

4. Projective resolutions of completely regular spaces. By a *projective resolution* of a c.r. space X we mean an $\mathscr{A}$-projective resolution of X which is an object of $\mathscr{CR}$, where $\mathscr{A}$ is the class of all perfect mappings of $\mathscr{CR}$.

In order to assure some good properties of projective resolutions (see § 1), we shall check conditions (1) and (2) for $\mathscr{A}$.

Condition (1) obviously holds. To prove (2) note that

(10) *$\mathscr{A}$-injections are one-to-one.*

To prove this, let $h\colon Z \to Y$ be an $\mathscr{A}$-injection. Suppose, on the contrary, that for some distinct points a and b we have $h(a) = h(b)$. Let R be an equivalence relation on Z the equivalence classes of which are $\{a, b\}$ and single points. Take the decomposition $Z \to Z/R \to Y$ of h. By the relation R just defined, the quotient space Z/R is c.r., so that the decomposition may be performed in a "sufficiently complete" $\mathscr{CR}$. Clearly, the quotient mapping is perfect (in virtue of Henriksen-Isbell characterization of perfect mappings, it suffices to check that $Z \to Z/R$ is closed). Now, since h is an $\mathscr{A}$-injection, $Z \to Z/R$ is an isomorphism. We have a contradiction.

Having (10), condition (2) is obvious.

Before the construction of projective resolutions for c.r. spaces, let us note a fact concerning embeddings.

(11) *A dense embedding into a compact c.r. projective space is the Čech-Stone embedding.*

For let $i\colon X \to P$ be an embedding of X as a dense subset $i(X)$ in a projective compact c.r. space P. The induced mapping $\beta X \to P$ transforms $\beta X - \beta_X(X)$ into $P - i(X)$. Since P is projective, there exists a mapping filling up the diagram

$$\begin{tikzcd} P \arrow[d] \arrow[dr, dashed] & \\ P & \beta X \arrow[l] \end{tikzcd}$$

where $P \to P$ is the identity. By the property of $\beta X \to P$ just mentioned, $P \to \beta X$ transforms $i(X)$ into X. Thus, the compactification $X \to P$ majorizes the Čech-Stone compactification $X \overset{\beta_X}{\to} \beta X$, hence it is isomorphic with β_X.

1.4. *Each completely regular space admits a projective resolution which is a strong semi-monomorphism.*

Let X be a c.r. space. To construct a projective resolution of X, we take first $\beta_X\colon X \to \beta X$ and $a^{\beta X}\colon a\beta X \to \beta X$, the projective resolution of βX in $\mathscr{CCR}$. Now take a uniformization

$$\begin{tikzcd} Z \arrow[r, dashed] \arrow[d, dashed] & a\beta X \arrow[d, "a^{\beta X}"] \\ X \arrow[r, "\beta_X"'] & \beta X \end{tikzcd}$$

of β_X and $a^{\beta X}$.

We shall show that $Z \to X$, the inverse image $\beta_X^{-1}(a^{\beta X})$ of the (compact) projective resolution of βX, is the desired projective resolution of X.

Note first that $Z \to X$ is onto and $Z \to a\beta X$ is an embedding by (6). Moreover, $Z \to a\beta X$ is an embedding onto a dense subset of $a\beta X$, for the closure of the image of $Z \to a\beta X$ is transformed onto βX (in virtue of the fact that $Z \to X$ is onto) and $\alpha^{\beta X}$: $\alpha\beta X \to \beta X$ is irreducible. Since $\alpha\beta X$ is projective in $\mathscr{CCR}$, $Z \to \alpha\beta X$ is a Čech-Stone embedding, by (11). This embedding will be denoted also by β_Z: $Z \to \beta Z$.

Now, the uniformization diagram is the Čech-Stone diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & \beta Z \arrow[d] \\ X \arrow[r] & \beta X \end{tikzcd}$$

Since $Z \to \beta Z$ is the inverse image of $X \to \beta X$ (also in the usual sense), $Z \to X$ is a perfect mapping.

The space Z is projective. To show this (8), let $W \to Z$ be a perfect mapping. Since $\beta Z = \alpha\beta X$ is projective in $\mathscr{CCR}$, $\beta W \to \beta Z$ is a retraction, by 2.1. A right inverse $\beta Z \to \beta W$ of $\beta W \to \beta Z$ induces a right inverse $Z \to W$ of $W \to Z$, according to the fact that $\beta W \to \beta Z$ transforms $\beta W - \beta_W(W)$ into $\beta Z - \beta_Z(Z)$, which is an induced mapping of a perfect one. Thus $W \to Z$ is a retraction, and, by 3.1, Z is projective.

Finally, we prove that $Z \to X$ is a strong semi-monomorphism. To prove this, let $Z \xrightarrow{q} Z$ be such that $Z \xrightarrow{q} Z \to X = Z \to X$. Consider the diagram

$$\begin{tikzcd} & Z \arrow[rr] \arrow[dl, "q"'] \arrow[dd] & & \beta Z \arrow[dl, dashed] \arrow[dd] \\ Z \arrow[rr] \arrow[dr] & & \beta Z \arrow[dr] & \\ & X \arrow[rr] & & \beta X \end{tikzcd}$$

consisting of two preceding diagrams joined by $Z \xrightarrow{q} Z$. Mapping $\beta Z \to \beta Z$ is the induced, in the Čech-Stone compactification, mapping of $Z \xrightarrow{q} Z \to \beta Z$. After a cancellation by (an epimorphism) $Z \to \beta Z$, we get the commutativity of the right triangle of the diagram. Now, since $\beta Z = \alpha\beta X \to \beta X$ is a strong semi-monomorphism, $\beta Z \to \beta Z$ is the identity. Hence, $Z \xrightarrow{q} Z \to \beta Z = Z \to \beta Z$. After a cancellation by (a monomorphism) $Z \to \beta Z$, $Z \xrightarrow{q} Z$ is the identity.

We know, by 1.1 and 1.2, that facts which we have proved imply that $Z \to X$ is the, unique up to an isomorphism, projective c. r. resolution of X.

Then, we may denote $Z \to X$ by α^X: $\alpha X \to X$.

(8) This may be proved also "topologically": Z is a dense subset of an e. d. space, hence it is e. d., hence projective, by 3.3.

By the way, we have proved that the operations α and β commute (see e.g. [7]). More precisely, the diagram

$$\begin{tikzcd} \alpha X \arrow[r, "\beta_{\alpha X}"] \arrow[d, "\alpha_X"'] & \alpha\beta X \arrow[d, "\alpha_{\beta X}"] \\ X \arrow[r, "\beta_X"'] & \beta X \end{tikzcd}$$

commutes.

5. A functorial character of the operation α. Let us notice (see § 1) that the operation α is functorial iff, in diagram (4), $Y \to \alpha X$ is uniquely determined by $Y \xrightarrow{f} X$.

5.1 (Henriksen and Jerison [6]). *In* $\mathscr{CCR}$ *mapping* $Y \to \alpha X$ *is uniquely determined by* $Y \xrightarrow{f} X$ *iff* $\overline{f^{-1}(\operatorname{Int} A)} = \overline{\operatorname{Int} f^{-1}(A)}$ *for each regularly closed subset* A *of* X.

Recall that A is *regularly closed* in X if A is the closure of the interior of A in X. Note that the inclusion $\subset$ in the formula of 5.1, holds for each continuous mapping. Call the mappings in 5.1 the *Henriksen-Jerison mappings*.

Before we extend this result to $\mathscr{CR}$, we prove a lemma.

(12) *If* $f\colon Y \to X$ *is a Henriksen-Jerison mapping, then* $\beta f\colon \beta Y \to \beta X$ *is so.*

In fact, let A be a regularly closed subset of βX. Then $B = \beta_X^{-1}(A)$ is a regularly closed subset of X and $\operatorname{Int} B = \beta_X^{-1}(\operatorname{Int} A)$. Using these facts, we get

$$\begin{aligned} \beta_Y^{-1}\big(\overline{\beta f^{-1}(\operatorname{Int} A)}\big) &= \overline{\beta_Y^{-1}\big(\beta f^{-1}(\operatorname{Int} A)\big)} \\ &= \overline{f^{-1}\big(\beta_X^{-1}(\operatorname{Int} A)\big)} = \overline{f^{-1}\big(\operatorname{Int} \beta_X^{-1}(A)\big)} \\ &= \overline{f^{-1}(\operatorname{Int} B)} = \overline{\operatorname{Int} f^{-1}(B)} \\ &= \overline{\operatorname{Int} f^{-1}\big(\beta_X^{-1}(A)\big)} = \overline{\operatorname{Int} \beta_Y^{-1}\big(\beta f^{-1}(A)\big)} \\ &= \overline{\beta_Y^{-1}\big(\operatorname{Int} \beta f^{-1}(A)\big)} = \beta_Y^{-1}\big(\overline{\operatorname{Int} \beta f^{-1}(A)}\big). \end{aligned}$$

Thus, after a cancellation by β_Y^{-1}, βf is a Henriksen-Jerison mapping.

5.2. *The preceding result holds in* $\mathscr{CR}$, *too: mapping* $Y \to \alpha X$ *is uniquely determined by* $Y \xrightarrow{f} X$ *iff* f *is a Henriksen-Jerison mapping.*

The proof of one of the implications consists in a formal reduction to the compact case.

Let $f\colon Y \to X$ be a Henriksen-Jerison mapping. If $Y \to \alpha X$ fills up the diagram

$$\begin{tikzcd}
\alpha X \arrow[d] & \\
X & Y \arrow[l] \arrow[ul, dashed]
\end{tikzcd}$$

then $\beta Y \to \beta\alpha X = \alpha\beta X$, the induced mapping of it in the Čech-Stone compactification, fills up the diagram

$$\begin{tikzcd}
\alpha\beta X = \beta\alpha X \arrow[d] & \\
\beta X & \beta Y \arrow[l] \arrow[ul, dashed]
\end{tikzcd}$$

Clearly, if mappings $Y \not\rightrightarrows \alpha X$ are different each of the other, then $\beta Y \not\rightrightarrows \beta\alpha X$ are so. But, by (12), $\beta Y \to \beta X$ is a Henriksen-Jerison mapping. Hence there exists only one mapping $\beta Y \to \alpha\beta X$ filling up the second of the diagrams. Thus there exists only one mapping filling up the first one.

The proof of the converse implication is not formal. But the proof, which will be omitted, is literally the same as that of [6] for the compact case.

REFERENCES

[1] J. Flachsmeyer, *Topologische Projektivräume*, Mathematische Nachrichten 26 (1963), p. 57-66.

[2] P. Freyd, *Abelian categories*, New York, Evanston and London 1964.

[3] L. Gillman and M. Jerison, *Rings of continuous functions*, New York 1959.

[4] A. M. Gleason, *Projective topological spaces*, Illinois Journal of Mathematics 2 (1958), p. 482-489.

[5] M. Henriksen and J. R. Isbell, *Some properties of compactifications*, Duke Mathematical Journal 25 (1958), p. 83-106.

[6] M. Henriksen and M. Jerison, *Minimal projective extensions of compact spaces*, ibidem 32 (1965). p. 291-295.

[7] С. Илиадис, *Абсолюты хаусдорффовых пространств*, Доклады Академии Наук СССР 149.1 (1963), p. 22-25.

[8] В. И. Пономарев, *Об абсолюте топологических пространств*, ibidem 149.1 (1963), p. 26-28.

[9] J. Rainwater, *A note on projective resolutions*, Proceedings of the American Mathematical Society 10 (1959), p. 734-735.

INSTITUTE OF MATHEMATICS OF THE WROCŁAW UNIVERSITY

Reçu par la Rédaction le 31. 8. 1966