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CONCERNING THE SET OF RETRACTIONS

BY

KAROL BORSUK (WARSZAWA)

1. Let X and Y be topological spaces. We denote by Y^X the space of all maps (= continuous functions) of X into Y with the "compact open topology" (in the sense of Fox [3]). The retractions of X, i.e. the maps $f: X \to X$ satisfying the condition

$$ff = f, \tag{1.1}$$

constitute a subset of X^X. This set will be denoted by $\mathfrak{R}(X)$. It is clear that for homeomorphic spaces X and Y the sets $\mathfrak{R}(X)$ and $\mathfrak{R}(Y)$ are homeomorphic. One easily sees that for a metric locally compact space X the set $\mathfrak{R}(X)$ is closed in X^X.

In this note we give some remarks concerning the topological properties of the set $\mathfrak{R}(X)$.

2. A space Y is said to be *starlike* if it is homeomorphic to a subset X of a normed linear space H such that

(2.1) *If* $x \in X$ *and* $0 \leqslant t \leqslant 1$, *then* $tx \in X$.

It is evident that every starlike space is contractible in itself, but not conversely. Let us prove the following

(2.2) THEOREM. *If X is a starlike (compact) ANR-space, then $\mathfrak{R}(X)$ is contractible in itself.*

Proof. We can assume that X is a subset of a normed linear space H satisfying (2.1). Since contractible ANR-spaces are the same as AR-spaces, X is an AR-space and consequently there is a retraction

$$s: H \to X. \tag{2.3}$$

Let us set

$$h_t(x) = (1-t)x \quad \text{for every } 0 \leqslant t \leqslant 1 \text{ and } x \in H. \tag{2.4}$$

It follows by (2.1) and (2.4) that

$$h_t(X) \subset X \text{ for every } 0 \leqslant t \leqslant 1.$$

Moreover, if $0 \leqslant t < 1$, then h_t maps H onto itself topologically, and h_1 is the retraction of H to the point $0 \in H$. Setting, for every map $r \in \mathfrak{R}(X)$,

$$\varphi(r, t) = h_t r s h_t^{-1} \quad \text{for} \quad 0 \leqslant t < 1,$$

$$\varphi(r, 1) = h_1,$$

we get a function φ assigning to every point $(r, t) \in \mathfrak{R}(X) \times \langle 0, 1 \rangle$ a map $\varphi(r, t)\colon X \to X$. It is evident that $\varphi(r, t)$ depends continuously on $(r, t) \in \mathfrak{R}(X) \times \langle 0, 1)$. Moreover, if $(r, t) \in \mathfrak{R}(X) \times \langle 0, 1)$, then $\varphi(r, t)$ maps X onto the set $h_t r s h_t^{-1}(X) \subset h_t r(X) \subset h_t(X)$. Since X is compact, we infer by (2.4) that for t sufficiently close to 1 the set $h_t(X)$ lies in an arbitrarily small neighborhood of the point 0. It follows that the dependence of $\varphi(r, t)$ on (r, t) is continuous in the whole set $\mathfrak{R}(X) \times \langle 0, 1 \rangle$. Moreover, for $0 \leqslant t < 1$, we have

$$\varphi(r, t)[\varphi(r, t)] = h_t r s h_t^{-1} h_t r s h_t^{-1} = h_t r s r s h_t^{-1}.$$

Since the values of the map $r s h_t^{-1}$ belong to X, we infer that $s r s h_t^{-1} = r s h_t^{-1}$ and, consequently, that $\varphi(r, t = [\varphi(r, t)] = h_t r r s h_t^{-1} = h_t r s h_t^{-1} = \varphi(r, t)$. Thus we see that the map $\varphi(r, t)\colon X \to X$ satisfies equation (1.1), i.e. $\varphi(r, t) \in \mathfrak{R}(X)$ for every $(r, t) \in \mathfrak{R}(X) \times \langle 0, 1)$. Since $\varphi(r, 1) = h_1$ for every map $r \in \mathfrak{R}(X)$, we infer that φ is a homotopy contracting the set $\mathfrak{R}(X)$ in itself to the point h_1. Thus the proof of Theorem (2.2) is finished.

(2.5) PROBLEM. *Is it true that for every starlike ANR-space X the set $\mathfrak{R}(X)$ is locally connected?* (**P 625**)

This problem remains open already in the case when X is the 2-dimensional disk.

3. Let us observe that if we replace in Theorem (2.2) the hypothesis that X is a starlike ANR-space by the weaker one that X is an AR-space, then the theorem ceases to be true. In order to see it, let us consider a 2-dimensional irreducible AR-space, i.e. a 2-dimensional AR-space X which does not contain any 2-dimensional AR-space $X' \neq X$. The existence of such AR-spaces has been proved in [2]. Now let i denote the identity map of X. Evidently, $i \in \mathfrak{R}(X)$ and there exists an $\varepsilon > 0$ such that for every map $f \in X^X$ the inequality $\varrho(i, f) < \varepsilon$ implies that $\dim f(X) = 2$. In particular, if $f \in \mathfrak{R}(X)$ and $\varrho(i, f) < \varepsilon$, then $f(X)$ is a 2-dimensional AR-set lying in X, whence $f(X) = X$, which implies $f = i$. Thus i is an isolated point of the set $\mathfrak{R}(X)$. But $\mathfrak{R}(X)$ contains many maps, and thus $\mathfrak{R}(X)$ is not connected, hence also not contractible.

Let us show that already among polyhedra there exist 2-dimensional AR-sets X for which the set $\mathfrak{R}(X)$ is not connected. It is so in the case when X is the *dunce hat*, i.e. a set homeomorphic to the set D which we obtain from the disk given in the complex plane by the inequality $|z| \leqslant 1$, if we identify each real number $0 \leqslant t \leqslant 1$ with the complex number $e^{2\pi i t}$. It is known ([1], p. 144) that the polyhedron D is an AR-set. Now let us prove the following

(3.1) THEOREM. *The identity map i of the dunce hat D is an isolated point of the set $\mathfrak{R}(D)$.*

Proof. It is clear that D has a triangulation $\mathcal{T}$ such that for every point $a \in D$ there exists a homeomorphism h_a mapping the union $U(a)$ of all simplexes of $\mathcal{T}$ containing the point a (the star of a) onto a cone with the vertex $h_a(a)$ and with the basis which is either a circle, or the union of a circle and of one of its diameters, or the union of two disjoint circles S_1, S_2 and of a segment L joining them.

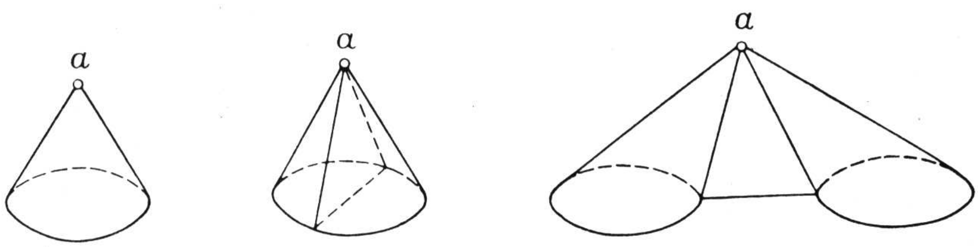

Fig. 1

In the first case we say that $U(a)$ is of the *first kind*, in the second — that it is of the *second kind*, in the third — that it is of the *third kind.* One easily sees that the points a for which the second case holds coincide with the points of D obtained by the identification of every real number $t \in (0, 1)$ with $e^{2\pi i t}$, and the unique point for which the third case holds is the point a_0 of D obtained by the identification of 0 with 1.

Let i denote the identity map of D. Using the well known results of H. Hopf and E. Pannwitz concerning stable points of a polyhedron ([4], p. 437), one readily sees that there exists a positive number ε such that for $f \in D^D$ the inequality $\varrho(f, i) < \varepsilon$ implies that $f(D)$ contains all stars $U(a)$ of the first and of the second kind. However, the star $U(a_0)$ of the third kind is not necessarily contained in $f(D)$. More exactly, h_{a_0} maps $U(a_0)$ onto the set which is the union of two cones C_1 and C_2 with the vertex $h_{a_0}(a_0)$ and the bases S_1 and S_2, respectively, and of a triangle Δ with the vertex $h_{a_0}(a_0)$ and the basis L. One easily sees that, for ε sufficiently small, the inequality $\varrho(f, i) < \varepsilon$ implies that

$$h_{a_0}^{-1}(S_1 \cup S_2 \cup L) \subset f(D),$$

yet it is possible that in the interior $\mathring{\Delta}$ of the triangle Δ there exists a point b such that the point $h_{a_0}^{-1}(b)$ does not belong to the set $f(D)$

(see [4], p. 447). In this last case, let us denote by r_0 a retraction of the set $h_{a_0}^{-1}(\Delta-(b))$ to the set $h_{a_0}^{-1}(\dot{\Delta})$, where $\dot{\Delta}$ denotes the boundary of the triangle Δ. Manifestly, if f is a retraction, then $f(x)=x$ for every point $x \epsilon D-h_{a_0}^{-1}(\overset{\circ}{\Delta})$. Setting

$$r(x)=\begin{cases} f(x) & \text{if} \quad f(x)\epsilon D-h_{a_0}^{-1}(\overset{\circ}{\Delta}), \\ r_0 f(x) & \text{if} \quad f(x)\epsilon h_{a_0}^{-1}(\overset{\circ}{\Delta}), \end{cases}$$

we get a retraction of D to the polyhedron $P_1=D-h_{a_0}^{-1}(\overset{\circ}{\Delta})$. Setting $P_2=h_{a_0}^{-1}(\Delta)$, we obtain a decomposition of D into the union of two polyhedrons P_1 and P_2 contractible in itself. However, this is impossible (see [1], p. 147). Hence none of the maps $f \epsilon D^D$ sufficiently close to the identity i is a retraction, and the proof of Theorem (3.1) is finished.

4. Now let us prove the following

(4.1) THEOREM. *In the Euclidean 3-space E^3 there exists a 2-dimensional AR-set A such that $\mathfrak{R}(A)$ has $2^{\aleph_0}$ components.*

Proof. Let $\{\Delta_n\}$ be a sequence of 3-dimensional simplexes in E^3 satisfying the following conditions:

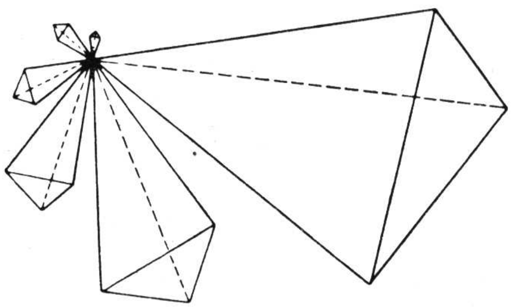

Fig. 2

1° There exists a point a being a vertex of every simplex Δ_n and such that $\Delta_i \frown \Delta_j=(a)$ for $i \neq j$.

2° $\lim\limits_{n=\infty} \delta(\Delta_n)=0$.

It follows by 1° and 2° that the set

$$Z=\bigcup_{n=1}^{\infty} \Delta_n$$

is an AR-set. Moreover, there exists a homeomorphism h_n mapping the dunce hat D onto a subset of Δ_n such that $a \epsilon h_n(D)$. Since D is an AR-set, there is a retraction $r_n\colon \Delta_n \to h_n(D)$. Setting

$$r(x)=r_n(x) \text{ for every point } x \epsilon \Delta_n, \qquad n=1,2,\ldots,$$

we get a retraction of the set Z to the set

$$A = \bigcup_{n=1}^{\infty} h_n(D). \tag{4.2}$$

It follows that A is an AR-set.

Now let us assign to every increasing sequence $\xi = \{n_k\}$ of natural numbers n_k the map r_ξ defined by the formulas:

$$r_\xi(x) = \begin{cases} x \text{ for every point } x \in h_n(D), \text{ if } n \text{ appears in the sequence } \xi, \\ a \text{ for every point } x \in h_n(D), \text{ if } n \text{ does not appear in the sequence } \xi. \end{cases}$$

Evidently, $r_\xi \in \mathfrak{R}(A)$ for every sequence ξ. Now let $\xi = \{n_k\}$ and $\xi' = \{n_k'\}$ be two different sequences. Since both sequences are increasing, there exists an index k such that either n_k does not appear in ξ', or n_k' does not appear in ξ. We can assume that the first possibility holds. Let B denote the "dunce hat" $h_{n_k}(D)$ and let r denote the retraction of A to B. Setting

$$\varphi(f) = rf/B \text{ for every map } f \in \mathfrak{R}(A),$$

we get a map $\varphi\colon \mathfrak{R}(A) \to \mathfrak{R}(B)$. Now let us observe that the map $s = \varphi(r_\xi)$ is the identity map of B and the map $s' = \varphi(r_{\xi'})$ is the retraction of B to the point a. By Theorem (3.1) s and s' belong to different components of the set $\mathfrak{R}(B)$, and, consequently, r_ξ and $r_{\xi'}$ belong to different components of $\mathfrak{R}(A)$. Since there exists $2^{\aleph_0}$ different sequences ξ, we infer that the set $\mathfrak{R}(A)$ contains at least $2^{\aleph_0}$ different components. But the power of the space A^A does not exceed $2^{\aleph_0}$ and we conclude that its subset $\mathfrak{R}(A)$ contains exactly $2^{\aleph_0}$ components. Thus the proof of Theorem (4.1) is finished.

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