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A CHARACTERIZATION OF HEREDITARILY DECOMPOSABLE SNAKE-LIKE CONTINUA

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The object of this note is to extend Bing's characterization of hereditarily decomposable snake-like continua to the non-metric setting. Specifically, we show that if M is a compact, connected, Hausdorff space and M is hereditarily decomposable, then M is snake-like if and only if M is hereditarily unicoherent and atriodic. The proof is modelled after Bing's proof of the same result for metric continua and uses a recent result of Gordh which shows that if M is a hereditarily decomposable, hereditarily unicoherent, atriodic (Hausdorff) continuum, then M admits a decomposition similar to the one produced by Bing in [1].

At the end of the paper an example is given to show that Gordh's decomposition need not have all the properties of Bing's.

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Definition 1. A *continuum* is a compact, connected, Hausdorff space.

Definition 2. A continuum M is said to be *irreducible between points p and q* in M if no proper subcontinuum of M contains both p and q. M is simply said to be *irreducible* if there are points p and q in M such that M is irreducible between p and q.

Definition 3. A point p of a space X is called a *separating point* of X if $X-\{p\}$ is disconnected.

Definition 4. A continuum is called *ordered* if it has exactly two non-separating points.

It is well known that if M is an ordered continuum, then M admits a total order such that its non-separating points are the first and last elements and such that the order topology on M is precisely the given topology (see, e.g., [5], p. 49-50). If M is an ordered continuum and $\leqslant$ is the order mentioned above, then we will use the notation $[x, y]$ to denote $\{z \in M : x \leqslant z \leqslant y\}$. Analogously we define (x, y), $(x, y]$ and $[x, y)$. Note that if a and b are the non-separating points of M and $a \leqslant b$, then $M = [a, b]$.

Definition 5. A continuum M is said to be *of type A'* if M is irreducible and there is a decomposition $\mathscr{D}$ of M into equivalence classes such that

(i) $\mathscr{D}$ is upper semicontinuous,

(ii) each $D \in \mathscr{D}$ is a continuum,

(iii) each $D \in \mathscr{D}$ has void interior in M,

(iv) the quotient space $M/\mathscr{D}$ is an ordered continuum.

M is said to be *hereditarily of type A'* if every non-degenerate subcontinuum of M (including M itself) is of type A'. Note that conditions (i)-(iv) are equivalent to saying that M admits a monotone, continuous function f from M onto an ordered continuum $[a, b]$ such that $f^{-1}(t)$ has void interior in M for each $t \in [a, b]$.

The reader is referred to [2] for basic facts concerning continua of type A' and hereditarily of type A'. In [2] Gordh gives the following characterization of continua hereditarily of type A':

THEOREM A. *A continuum M is hereditarily of type A' if and only if it is hereditarily decomposable, hereditarily unicoherent and atriodic.*

The proof that continua which are hereditarily decomposable, hereditarily unicoherent and atriodic are snake-like will rely heavily on the above-mentioned characterization of such continua.

Definition 6. A continuum M is said to be *unicoherent* if whenever M is realized as the union of subcontinua A and B, it follows that $A \cap B$ is connected. M is said to be *hereditarily unicoherent* if each of its subcontinua is unicoherent.

Definition 7. A continuum M is said to be *decomposable* if it can be written as the union of two proper subcontinua. M is said to be *hereditarily decomposable* if each of its (non-degenerate) subcontinua is decomposable.

Definition 8. A continuum T is called a *triod* if it contains a subcontinuum H such that $T - H$ can be written as the union of three disjoint non-void open sets. A continuum M is said to be *atriodic* if it contains no triods.

Definition 9. A finite collection $\mathscr{C}$ of sets is called a *chain* if the elements of $\mathscr{C}$ can be indexed $C_1, C_2, \ldots, C_n$ in such a way that $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$.

Definition 10. A continuum M is said to be *snake-like* if every open cover of M has an open refinement which covers M and is a chain (note that since M is compact, this is equivalent to saying that every finite open cover of M has an open refinement which covers M and is a chain).

We are now ready to begin proving the theorem mentioned in the introduction. This will be done in a series of steps. For the rest of the discussion let M denote a fixed hereditarily decomposable, hereditarily unicoherent, atriodic continuum and let $\mathscr{D}$ be a decomposition of M like the one in the definition of continua of type A'. Let f be the natural map of M onto the ordered continuum $M/\mathscr{D}$, which we denote by $[a, b]$.

Definition 11. A point $p \in M$ is said to have *property A* if every subcontinuum L of M which contains p is irreducible between p and some other point of L.

LEMMA 1. *M contains a point with property A.*

Proof. Let $\mathscr{H}$ denote the family of all subcontinua H of M with the following property: if L is any subcontinuum of M which properly contains H, then there is a point $q \in L$ such that no proper subcontinuum of L contains q and meets H.

Note that $\mathscr{H}$ is non-void since the continuum $f^{-1}(a)$ has this property. Now, let $\{H_a: a \in \Gamma\}$ be a maximal nest of elements of $\mathscr{H}$ and let $H = \bigcap \{H_a: a \in \Gamma\}$.

First we show that $H \in \mathscr{H}$. Indeed, let L be a subcontinuum of M which properly contains H. There must be an a such that H_a does not contain L (otherwise, $L \subset \bigcap \{H_a: a \in \Gamma\} = H$). Then the continuum $H_a \cup L$ properly contains H_a. Thus there is a point $q \in L \cup H_a$ such that no proper subcontinuum of $L \cup H_a$ contains q and meets H_a. But $q \in L$ (this implies that $q \notin H_a$) and L meets H_a. Therefore, it must be the case that $L = L \cup H_a$. Thus no proper subcontinuum of L contains q and meets H_a. But since $H \subset H_a$, this clearly implies that no proper subcontinuum of L contains q and meets H.

Next we prove that H is a point. For suppose that H were a non-degenerate continuum. Then, since M is hereditarily of type A', H would admit a monotone, continuous map g onto an ordered continuum $[c, d]$. Then $g^{-1}(c)$ would be a proper subcontinuum of H with the property that every subcontinuum L of H which properly contains $g^{-1}(c)$ contains a point q such that no proper subcontinuum of L contains q and meets $g^{-1}(c)$. It is now possible to give a proof similar to that of $H \in \mathscr{H}$ showing that $g^{-1}(c) \in \mathscr{H}$. But this violates the maximality of the nest $\{H_a: a \in \Gamma\}$.

Thus H must be a point, say $H = \{p\}$. Since $H \in \mathscr{H}$, it is clear that p has property A.

LEMMA 2. *If $\mathscr{C} = \{C_1, C_2, \ldots, C_n\}$ is a chain of open sets covering M and $p \in M$ has property A, then there is a chain $\mathscr{E} = \{E_1, E_2, \ldots, E_m\}$ of open sets covering M which refines $\mathscr{C}$ and such that $p \in E_1 - \mathrm{Cl}(E_2)$.*

The proof of this result is strictly analogous to Bing's proof of the same result in the metric setting (see [1], theorem 10, p. 659).

THEOREM 1. *M is snake-like.*

Proof. Let $\mathcal{U}$ be a finite open cover of M and suppose that $\mathcal{U}$ does not admit a finite open refinement which covers M and is a chain. Then there is a subcontinuum M' of M which is minimal with respect to this property for the fixed cover $\mathcal{U}$ (M' will be non-degenerate since each element of $\mathcal{U}$ contains non-degenerate continua). For notational convenience we assume (without loss of generality) that $M' = M$.

Let D be any element of $\mathcal{D}$ other than $f^{-1}(a)$ or $f^{-1}(b)$, say $D = f^{-1}(t)$. Define continua A and B by

$$A = \mathrm{Cl}\big(f^{-1}([a, t))\big)\cap D \quad \text{and} \quad B = \mathrm{Cl}\big(f^{-1}((t, b])\big)\cap D,$$

respectively. A and B are continua since f is monotone and M is hereditarily unicoherent. Moreover, since M is connected and D has the void interior in M, we must have $A\cap B \neq \emptyset$ and $A\cup B = D$.

Since M is hereditarily of type A', the continua A, B and $A\cup B$ are all of type A' (if any of these continua are degenerate, it will be clear how to choose the point p described below). A proof similar to that of lemma 1 will show that either A or B must contain a point p which has property A both with respect to itself and with respect to the continuum $A\cup B$. Suppose (without loss of generality) that A contains such a point p. Using the fact that M is atriodic, one can also choose p so that if L is any subcontinuum of $A\cup B$ which contains $\{p\}\cup B$, then $L = A\cup B$ (if this were not possible, then the continuum $\mathrm{Cl}\big(f^{-1}((t, b])\big)\cup A$ would be a triod). It can now be shown that p has property A with respect to the continua

$$H = \mathrm{Cl}\big(f^{-1}([a, t))\big) \quad \text{and} \quad K = \mathrm{Cl}\big(f^{-1}((t, b])\big)\cup A.$$

Since H and K are proper subcontinua of M, lemma 2 implies that they can be covered by chains of open sets $H_1, \ldots, H_n$ and $K_1, \ldots, K_m$, respectively, such that these chains refine $\mathcal{U}$ and $p\in\big(H_n-\mathrm{Cl}(H_{n-1})\big)\cap \cap\big(K_1-\mathrm{Cl}(K_2)\big)$.

Now, let N be a neighborhood of p whose closure is contained in $\big(H_n-\mathrm{Cl}(H_{n-1})\big)\cap\big(K_1-\mathrm{Cl}(K_2)\big)$. Since $\mathcal{D}$ is upper semi-continuous and $K_1\cup K_2 \cup \ldots \cup K_m$ contains $K = f^{-1}([t, b])$, there must be a $t_1 \leqslant t$, $t_1 \neq t$, such that

$$f^{-1}([t_1, b]) \subset K_1\cup K_2\cup \ldots \cup K_m.$$

Also, since A has void interior in $\mathrm{Cl}\big(f^{-1}([a, t))\big)$, there must be a $t_2\leqslant t$, $t_2 \neq t$, such that

$$N-f^{-1}([t_2, b]) \neq \emptyset.$$

Let $t_0 = \max\{t_1, t_2\}$ and let $U = N-f^{-1}([t_0, b])$.

Since M is irreducible, $M-U$ must be disconnected (it may be necessary to choose N so that $f^{-1}(a)\cap N = f^{-1}(b)\cap N = \emptyset$) and the continuum $f^{-1}([t_0, b])$ will be a component of the compact set $M-U$. Since $f^{-1}([t_0, t])$ is contained in the open set $K_1 \cup \ldots \cup K_m$, there must be mutually separated sets H' and K' such that

$$H' \cup K' = M-U \quad \text{and} \quad f^{-1}([t_0, t]) \subset K' \subset K_1 \cup \ldots \cup K_m.$$

Thus $K \subset K' \subset K_1 \cup \ldots \cup K_m$. Intersecting H' and K' with $M-N$, we obtain mutually separated sets H'' and K'' such that

$$H'' \cup K'' = M-N \quad \text{and} \quad K-N \subset K'' \subset K_1 \cup \ldots \cup K_m.$$

Now H'' and K'' are closed. Thus $H'' \cup N = M-K''$ and $K'' \cup N = M-H''$ are open. It is then straightforward to check that $H'' \cap H_1$, $H'' \cap H_2, \ldots, H'' \cap H_{n-1}$, $(H'' \cap H_n) \cup N$, $(K'' \cap K_1) \cup N$, $K'' \cap K_2$, $K'' \cap K_3$, $\ldots, K'' \cap K_m$ is a chain of open sets which covers M and refines $\mathcal{U}$.

It is not difficult to show that any snake-like continuum must be hereditarily unicoherent and atriodic. Thus we have proved the following

THEOREM 2. *A hereditarily decomposable continuum is snake-like if and only if it is atriodic and hereditarily unicoherent.*

It is not difficult to show that snake-like continua have the fixed-point property (the basic trick was produced by Hamilton in [4]). In fact, one can show the following(¹)

THEOREM B. *If M is a snake-like continuum, then, given any continuum X and any two continuous functions f and g from X into M, if either f or g is onto, then there is an $x \in X$ such that $f(x) = g(x)$.*

Thus continua hereditarily of type A' have the fixed-point property. This fact generalizes a theorem in [3] which states that such continua have the fixed point property for homeomorphisms.

Bing's proof of theorem 1 in the metric setting makes use of the fact that if M is a metrizable continuum of type A', then (using the same notation as above), for uncountably many $t \in [a, b]$, the set-valued mapping $s \to f^{-1}(s)$ is continuous at t. Points t for which this is true are special cases of t's for which $f^{-1}(t)$ is what Kuratowski calls a *layer of cohesion* of M.

Definition 12 (with notation as above). A set $f^{-1}(t)$ is called a *layer of cohesion* of M if

$$f^{-1}(t) = \mathrm{Cl}\big(f^{-1}([a, t))\big) \cap \mathrm{Cl}\big(f^{-1}((t, b])\big).$$

Kuratowski has shown (see [6], Remarks, p. 201) that if M is a metric continuum of type A', then, for all but countably many $t \in [a, b]$, $f^{-1}(t)$ is a layer of cohesion of M. We will give an example which shows that this need not be the case in the non-metric setting.

(¹) W. Holsztyński has conjectured that this theorem characterizes snake-like continua among all other continua.

Let Y be the space whose underlying set consists of two points x_t and y_t for each $t \in (0, 1)$ and, in addition, contains points y_0 and x_1. We linearly order Y as follows:

For every $t \in (0, 1)$, set $x_t < y_t$ and if $t_1 < t_2$, then $x_{t_1} < y_{t_1} < x_{t_2} < y_{t_2}$.

It is not difficult to verify that Y with the order topology is a compact T_2-space (Y can be thought of as a Cantor set which contains nothing but end points). Consider the space $X = Y \times I$, where I denotes the closed unit interval of real numbers. Now perform the following identifications on X: for all rational $t \in (0, 1)$, identify the two points $(x_t, 1)$ and $(y_t, 1)$ in X and, for all irrational $t \in (0, 1)$, identify the two points $(x_t, 0)$ and $(y_t, 0)$. The resulting quotient space will be a continuum hereditarily of type A' which contains no layers of cohesion.

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