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ON THE CONTINUA WHICH ARE CANTOR HOMOGENEOUS OR ARCWISE HOMOGENEOUS

BY

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1. Introduction. We shall consider only metric spaces and by a *Cantor set* we shall mean any space homeomorphic to the usual Cantor discontinuum $C \subset E^1$. A space X containing a Cantor set will be called *Cantor homogeneous* if for any two Cantor sets $A, B \subset X$ there is an autohomeomorphism h of X mapping A onto B. It is well known that closed manifolds of dimension $\leqslant 2$ are Cantor homogeneous but n-manifolds with $n > 2$ as well as the Hilbert cube are not, since they contain wild Cantor sets (cf. [3] and [4]). The authors do not know other continua which are Cantor homogeneous; moreover, it is easy to see that the universal Menger curve M_1^3 (whose homogeneity has been proved by Anderson in [1]) is not Cantor homogeneous, and we know from the private communication from M. Bestvina who proved the homogeneity of the remaining universal Menger continua M_n^{2n+1} (see [2]) that they also are not Cantor homogeneous. Thus the problem arises to describe all continua which are Cantor homogeneous. (**P 1377**)

In Section 2 we shall prove some properties of Cantor homogeneous continua, in particular we show that they are n-homogeneous for any $n = 1, 2, \ldots$ and locally connected.

Related problems concern the arcwise homogeneity, where a space X containing an arc is called *arcwise homogeneous* if for any two arcs $K, L \subset X$ there is an autohomeomorphism h of X such that $h(K) = L$. In the class of arcwise connected continua the same examples as before are known to be or not to be arcwise homogeneous, and we do not know if for this class these two kinds of homogeneity coincide. This is not the case for non-arcwise connected continua, since, as we show in Section 3, solenoids are arcwise homogeneous and are not Cantor homogeneous.

Observe here that any arcwise connected and arcwise homogeneous space is 2-homogeneous, because any homeomorphism of arcs must preserve the end-points. It follows from a theorem of Ungar (cf. [10]) that any continuum satisfying these assumptions is strongly 2-homogeneous and locally connected.

However we do not know if such a continuum must be n-homogeneous for $n > 2$.

All known examples of arcwise connected and Cantor (or arcwise) homogeneous continua different from S^1 satisfy the following stronger condition (cf. [7]): A space X containing a Cantor set (an arc) will be called *strongly Cantor* (*arcwise*) *homogeneous* if for any homeomorphism h_0: $A \to B$, where $A, B \subset X$ are Cantor sets (arcs), there is an autohomeomorphism h of X such that $h|_A = h_0$. It is easy to see that S^1 is strongly arcwise homogeneous, but it is not strongly Cantor homogeneous. We do not know if continua different from S^1 which are Cantor homogeneous are also strongly Cantor homogeneous and if any arcwise homogeneous continuum is also strongly arcwise homogeneous.

In Section 3 we shall prove that any arcwise connected continuum different from S^1 which is strongly arcwise homogeneous is also strongly Cantor homogeneous. We do not know whether the other implication holds.

2. Continua which are Cantor homogeneous. The following easy proposition shows that the assumption of connectivity in the next results can be replaced by a weaker one:

2.1. PROPOSITION. *Any Cantor homogeneous compact space X without isolated points is connected.*

Proof. Since the Cantor discontinuum $C \subset E^1$ is not Cantor homogeneous, X contains a non-degenerate component P. There are two Cantor sets $A, B \subset X$, where

$$A \subset P \quad \text{and} \quad B \cap P \neq \emptyset \neq B \cap (X \setminus P).$$

Evidently, no autohomeomorphism of X maps A onto B.

The next Propositions 2.2 and 2.3 follow from Theorem 2.7 below, since Ungar has proved in [10] that the 2-homogeneity of a continuum implies local connectedness. However, we include their independent and simpler proofs.

2.2. PROPOSITION. *Any Cantor homogeneous continuum X is homogeneous.*

Proof. Denote by H the group of all autohomeomorphisms of X and consider the orbit Hx_0 of a point $x_0 \in X$. Since the group H is topologically complete, Hx_0 is an analytic subset of X. Assume that $Hx_0 \neq X$. If Hx_0 is uncountable, then it contains a Cantor set A (cf. [5], p. 479). Let $B \subset X$ be any Cantor set such that $B \setminus Hx_0 \neq \emptyset$. Since every autohomeomorphism of X maps any orbit onto itself, we obtain a contradiction with the Cantor homogeneity of X.

If Hx_0 is countable, then there are two Cantor sets $A, B \subset X$ such that $A \cap Hx_0 = \emptyset$ and $x_0 \in B$, which also yields a contradiction.

2.3. PROPOSITION. *Any Cantor homogeneous continuum is locally connected.*

Proof. First, observe that a continuum X is locally connected iff for any Cantor set $A \subset X$ and for any $x \in A$ the following condition is satisfied:

(*) For each neighborhood U of x in X we have

$$\{x\} \subsetneq C_x \cap A,$$

where C_x denotes the component of U containing x.

Indeed, it suffices to observe that if X is not locally connected in x, then there is a Cantor set $A \subset X$ containing x such that (*) is not satisfied. In fact, by the absence of the local connectivity of X in x, there are a closed neighborhood U of x and a sequence $\{C_n\}_{n=1}^{\infty}$ of components of U such that $x \in \mathrm{Li}\, C_n$. Since these components are non-degenerate, each C_n contains a Cantor set A_n such that

$$\operatorname{diam} A_n < 1/n \quad \text{and} \quad \{x\} = \operatorname{Lim} A_n.$$

Then

$$A = \bigcup_{n=1}^{\infty} A_n \cup \{x\}$$

is a Cantor set containing x such that (*) is not satisfied.

Now, assume that X is a Cantor homogeneous continuum. To prove the assertion, it suffices to find a Cantor set $A \subset X$ such that for each $x \in A$ the condition (*) is satisfied. For this purpose we shall construct a sequence $\{F_{i_1 \dots i_k}\}$, where $i_j = 0, 1$ and

$$\operatorname{diam} F_{i_1 \dots i_k} < 1/k \quad \text{for } k = 1, 2, \dots,$$

of compact subsets of X as in the usual construction of the Cantor discontinuum. To find F_0 and F_1 choose two open non-empty subsets G_0, G_1 of X such that

$$\operatorname{diam} G_i < 1 \quad \text{and} \quad \bar{G}_0 \cap \bar{G}_1 = \emptyset.$$

Choose a component C_i of G_i and define $F_i = \bar{C}_i$. To find F_{00} and F_{01} choose two open sets $H_0, H_1 \subset X$ such that

$$\operatorname{diam} H_i < 1/2, \quad \bar{H}_0 \cap \bar{H}_1 = \emptyset \quad \text{and} \quad H_i \cap C_0 \neq \emptyset.$$

Choose any component D_i of $H_i \cap C_0$ and define $F_{0i} = \bar{D}_i$. The construction is extended in a natural way and the desired Cantor set A is defined by the formula

$$A = \bigcap_{k=1}^{\infty} \bigcup \{F_{i_1 \dots i_k} \colon i_j = 0, 1\}.$$

Let (X, d) be a compact space and consider the space 2^X consisting of closed non-empty subsets of X with Hausdorff metric D. Denote by $\mathscr{C}$ the subspace of 2^X consisting of Cantor sets, and by $H(X)$ the group of all autohomeomorphisms of X. The following proposition is a version of the well-known Effros theorem concerning homogeneous spaces (cf., e.g., [10]), applied to the space $\mathscr{C}$.

2.4. PROPOSITION. *If (X, d) is a compact, Cantor homogeneous space, then for each $\varepsilon > 0$ and each Cantor set $C \in \mathscr{C}$ there exists a $\delta > 0$ such that: if $C' \in \mathscr{C}$ and $D(C, C') < \delta$, then there exists an autohomeomorphism $h \in H(X)$ such that*

$$h(C) = C' \quad \text{and} \quad d(x, h(x)) < \varepsilon \text{ for any } x \in X.$$

Proof. First notice that the space $\mathscr{C}$ is topologically complete. Indeed, let

$$F_n = \{A \in 2^X\colon \text{ there is a connected set } B \subset A \text{ with } \operatorname{diam} B \geqslant 1/n\},$$

$$H_n = \{A \in 2^X\colon \text{ there is } p \in A \text{ such that } A \cap \{x \in X\colon d(x, p) < 1/n\} = \{p\}\}.$$

Then both F_n and H_n are closed subsets of 2^X and

$$\mathscr{C} = 2^X \setminus \Big(\bigcup_{n=1}^{\infty} F_n \cup \bigcup_{n=1}^{\infty} H_n\Big).$$

Thus $\mathscr{C}$ is a G_δ-subset of the compact space 2^X.

Since (X, d) is Cantor homogeneous, the group $(H(X), \hat{d})$ acts transitively on $\mathscr{C}$, where $\hat{d}$ is the usual supremum metric, and one sees that all the assumptions of the Effros theorem are satisfied. Applying this theorem we obtain the assertion.

The next lemma implies the main result of this section. For each set A let

$$F_n(A) = \{(x_1, \ldots, x_n) \in A^n\colon x_i \neq x_j \text{ for } i \neq j\}.$$

Recall that a space X is *strongly locally n-homogeneous* if for any $(x_1, \ldots, x_n) \in F_n(X)$ there is an $\varepsilon > 0$ such that if

$$(y_1, \ldots, y_n) \in F_n(X) \quad \text{and} \quad d(x_i, y_i) < \varepsilon \text{ for } i \leqslant n,$$

then there is an $h \in H(X)$ such that $h(x_i) = y_i$ (cf. [10]).

2.5. LEMMA. *If X is a Cantor homogeneous continuum, then X is strongly locally n-homogeneous for each $n = 1, 2, \ldots$*

Proof. Our proof is based on a corollary to Theorem 1 of Mycielski's paper [8]. For convenience of the reader we shall refer to the paper [4], where the needed corollary is given explicitly.

First, notice that $F_n(X)$ is a *Polish space* (i.e., topologically complete and separable) and $X^n \setminus F_n(X)$ is of the first category (of Baire) in X^n. For any $\hat{x} = (x_1, \ldots, x_n) \in F_n(X)$ denote by $O(\hat{x})$ the orbit of $\hat{x}$ under the action of the group $H(X)$ on $F_n(X)$; i.e.,

$$O(\hat{x}) = \{h(\hat{x}) = (h(x_1), \ldots, h(x_n))\colon h \in H(X)\}.$$

Then $O(\hat{x})$ is an analytic subset of $F_n(X)$, and therefore it has the Baire property (cf. [5], p. 482). Notice that the strong local n-homogeneity of X means that each orbit $O(\hat{x})$ is open in $F_n(X)$.

Assume that there is an $\hat{x} \in F_n(X)$ such that $O(\hat{x})$ is of the first category in X^n. In virtue of [4], Lemma 2.2, there is a Cantor set $C \subset X$ such that

$$F_n(C) \cap O(\hat{x}) = \emptyset.$$

On the other hand, there is a Cantor set $C' \subset X$ containing the "coordinates" $x_1, \ldots, x_n$ of $\hat{x}$. The existence of a homeomorphism $h \in H(X)$ mapping C onto C' gives a contradiction, because $O(\hat{x})$ is the orbit of $\hat{x}$ in $F_n(X)$.

Thus any set $O(\hat{x})$ is of the second category in X^n, and therefore in $F_n(X)$. Since it has the Baire property, there is an open non-empty set U in $F_n(X)$ such that $U \setminus O(\hat{x})$ is of the first category. Observe that $U \subset O(\hat{x})$. Indeed, in the other case there is a $\hat{y} \in U \setminus O(\hat{x})$. Thus the orbit $O(\hat{y})$ intersects U on the set of the first category. On the other hand, there is an open subset V of $F_n(X)$ such that $V \setminus O(\hat{y})$ is of the first category. This contradicts the fact that there is an autohomeomorphism of $F_n(X)$ mapping $\hat{y}$ onto a point $\hat{z} \in V \cap O(\hat{y})$. Consequently, $U \subset O(\hat{x})$, and therefore the set

$$O(\hat{x}) = \bigcup \{h(U) \colon h \in H(X)\}$$

is open in $F_n(X)$, which completes the proof.

2.6. Remark. Our proof of Lemma 2.5 shows in fact the following equivalence: Let X be a Polish space and H a topological group which is also a Polish space acting continuously on X. Then each orbit is open in X iff each orbit is of the second category in X.

2.7. THEOREM. *If X is a Cantor homogeneous continuum which is (topologically) different from S^1, then X is strongly n-homogeneous for each $n = 1, 2, \ldots$*

Proof. First notice that the strong local 1-homogeneity and the connectedness of X imply the homogeneity of X.

Assume that there is a finite set which disconnects X. Then X contains a point which locally disconnects X, and therefore each $x \in X$ locally disconnects X. By Whyburn's theorem [12], any point locally disconnecting X (except of at most countably many of them) is a point of order 2 (in the sense of Menger–Urysohn). By the homogeneity of X, each $x \in X$ is a point of order 2, which implies that

$$X \underset{\text{top}}{=} S^1$$

(cf. [6], p. 294).

Thus no finite set disconnects X, which implies, by Lemma 3.9 of [10], that $F_n(X)$ is connected for $n = 1, 2, \ldots$ This fact and Lemma 2.5 imply that X is strongly n-homogeneous.

By Theorem 3.3 of [11] we have

2.8. COROLLARY. *Any Cantor homogeneous continuum is countable dense homogeneous.*

2.9. COROLLARY. *Let $X = X_1 \times \ldots \times X_n$, where $n \geqslant 1$ and each X_i is a Peano curve. Then X is Cantor homogeneous iff X is homeomorphic either with S^1 or with the torus $S^1 \times S^1$.*

Proof. If $n = 1$, then the homogeneity of X and Anderson's theorem [1] imply that $X = X_1$ is homeomorphic either with S^1 or with M_1^3. Since M_1^3 is not Cantor homogeneous, we have

$$X \underset{\text{top}}{=} S^1.$$

If $n > 1$, then the 2-homogeneity of X and a corollary given in [9] (p. 345) imply that

$$X \underset{\text{top}}{=} S^1 \times \ldots \times S^1.$$

Since no n-manifold with $n \geqslant 3$ is Cantor homogeneous, we infer that

$$X \underset{\text{top}}{=} S^1 \times S^1.$$

2.10. COROLLARY. *Let $X = X_1 \times \ldots \times X_n$, where $n \geqslant 1$ and each X_i is a compact, connected ANR-space of dimension $\leqslant 2$. Then X is Cantor homogeneous iff X is a closed manifold of dimension $\leqslant 2$.*

Proof. The homogeneity of X and Theorem 2 in [9] imply that X is a closed manifold. As before, we have $\dim X \leqslant 2$.

3. Arcwise homogeneity of continua and a connection with Cantor homogeneity. First, let us show the following

3.1. EXAMPLE. *Each solenoid X is strongly arcwise homogeneous and it is not Cantor homogeneous.*

Proof. A space X is a *solenoid* provided there exists a sequence $\{p_n\}$ of positive integers greater than one such that

$$X = \varprojlim\{X_n, f_n\},$$

where

$$X_n = S^1 = \{z \in C \colon |z| = 1\}$$

and the bonding map $f_n\colon X_{n+1} \to X_n$ is defined by $f_n(z) = z^{p_n}$. It is well known that each solenoid is an indecomposable continuum whose composants, being continuous one-to-one images of the real line E^1, can be ordered in some way. Now, given two arcs $K_1 = a_1 b_1$, $K_2 = a_2 b_2$ in X and a homeomorphism

$$h_0\colon (K_1, a_1) \to (K_2, a_2)$$

we can assume that $a_1 = a_2$ and that the ordering of K_i from a_i to b_i coincides with the ordering of the composant containing a_1. Indeed, using the fact that each solenoid is a topological group (with the neutral element e), for any arc $K = ab \subset X$ one can find two autoisomorphisms f, g of X such that $f(a) = e$ and

$$g(f(K)) = f(K) = ef(b), \quad g(e) = f(b), \quad g(f(b)) = e.$$

Next, if $K_1 = a_1 b_1$, $K_2 = a_1 b_2$, then the arc $K_1 \cup K_2$ is contained in an open subset U of X homeomorphic with the product $]0, 1[\times C$, where C is the Cantor discontinuum. The homeomorphism h_0 can be extended to an autohomeomorphism h of X which is the identity outside U.

Considering two Cantor sets $A, B \subset X$, where A is contained in a composant of X and B intersects at least two composants, it is evident that X is not Cantor homogeneous (cf. Proposition 2.3).

As noticed in the Introduction, in the class of arcwise connected continua such an example does not exist, and we shall prove the following

3.2. THEOREM. *Each arcwise connected continuum different (topologically) from S^1 which is strongly arcwise homogeneous is also strongly Cantor homogeneous.*

First notice the following

3.3. PROPOSITION. *Each arcwise connected and arcwise homogeneous continuum X is strongly 2-homogeneous, locally connected and, in the case where X is (topologically) different from S^1, it contains no locally disconnecting points.*

Proof. As noticed in the Introduction, the strong 2-homogeneity and the local connectivity of X follow from Ungar's paper [10]. The fact that X contains no locally disconnecting points follows from Whyburn's theorem [12] and the homogeneity of X, similarly as in the proof of Theorem 2.7.

3.4. COROLLARY. *If X satisfies the assumptions of Proposition* 3.3, *then each Cantor set $A \subset X$ is contained in an arc.*

Proof. This follows from Whyburn's theorem [13] which says that if X is a locally compact, locally connected space with no locally disconnecting points, A is a compact totally disconnected subset of X and $p, q \in A$ with $p \neq q$, then there exists an arc $L \subset X$ with end-points p, q containing A.

To prove Theorem 3.2 we need something more, namely, given a linear ordering $<$ of A which conforms to the topology of A, we should find an arc $L \subset X$ containing A such that the ordering $<$ coincides with the ordering of A induced by the natural ordering of the arc L.

For this purpose, let us prove first the lemmas:

3.5. LEMMA. *Let X be a locally compact, locally connected and connected space such that no region (i.e., open and connected set) in X is disconnected by an arc. Let $K \subset X$ be an arc with end-points p, q and with a linear ordering $<$ from p to q and let $A \subset \mathring{K}$ be a compact, totally disconnected set. Given any subarc $J = ab$ of K with a given ordering $\prec$ from a to b such that*

$$\emptyset \neq J \cap A \subset \mathring{J},$$

there are an arc $L = pq \subset X$ containing A and a subarc $J' = a'b'$ of L such that the subarc pa' of L does not intersect A, and

$$J \cap A \subset \mathring{J}' \subset J' \subset J \quad \text{and} \quad a' \prec b' \text{ on } J.$$

Proof. We shall assume that $b < a$ on the arc K, because in the other case the proof is similar, but simpler. Since no arc disconnects X, the set $G = X \backslash K$ is a region in X. Since the set of the points of K which are accessible from G is dense in K, there are two points $p' \in K$, $a' \in J$ and two arcs J_1, J_2 such that

$$p' \in \dot{J}_1, \quad a' \in \dot{J}_2, \quad J_1 \backslash \{p'\} \subset G, \quad J_2 \backslash \{a'\} \subset G$$

and the subarcs pp' and $a'a$ of K do not intersect A. Join the end-points of the arcs J_1, J_2 belonging to G by an arc $J \subset G$. Evidently, the union $J_1 \cup J \cup J_2$ contains an arc J' joining p' and a' and such that $\mathring{J}' \subset G$.

Now, let us enlarge the arc J' to an arc J'' such that

$$p \in J'', \quad J'' \supset J', \quad J'' \backslash J' \subset K \quad \text{and} \quad \mathring{J}'' \supset A \cap pa',$$

where pa' denotes the suitable subarc of the arc K. Find a subarc K' of K such that

$$q \in \dot{K}', \quad a' \notin K' \quad \text{and} \quad K' \supset A \cap qa',$$

where $qa' \subset K$. Since no region in X is disconnected by an arc, the set

$$H = X \backslash (J'' \cup K')$$

is a region in X. We can assume that the end-points of the arcs J'' and K' different from p and q are accessible from H, respectively, by the arcs L_1 and L_2. Join the end-points of these arcs lying in H by an arc $L_3 \subset H$. It is clear that the union $J'' \cup L_1 \cup L_3 \cup L_2 \cup K'$ contains the desired arc L.

3.6. LEMMA. *Let X satisfy the assumptions of Lemma 3.5, let $A \subset X$ be a compact, totally disconnected set containing more than one point and let $<$ be a linear ordering of A which conforms to the topology of A. Then there is an arc $L \subset X$ containing A such that the ordering $<$ coincides with the ordering of A induced by the natural ordering of L.*

Proof. First notice that the assumptions on X imply that X contains no locally disconnecting points, and therefore, by [13], there is an arc $L_0 = pq \subset X$ such that $A \subset \mathring{L}_0$ and that the first and the last points of A in the ordering $<$ coincide, respectively, with the first and the last points of A in the natural ordering of L_0 from p to q. We can easily find a sequence

$$\mathscr{G}_1 = \{G_{11}, \ldots, G_{1k_1}\}, \ \mathscr{G}_2 = \{G_{21}, \ldots, G_{2k_2}\}, \ldots$$

such that G_{ij} is a region in X intersecting A, $\bar{G}_{ij}$ is compact, $\bar{G}_{ij} \cap \bar{G}_{ij'} = \emptyset$ for $j \neq j'$, $\bigcup \{G_{ij} \colon j \leqslant k_i\}$ covers A for any i, $\operatorname{diam} G_{ij} < 1/i$ for any $j \leqslant k_i$ and the covering

$$\bar{\mathscr{G}}_{i+1} = \{\bar{G}_{i+1\,1}, \ldots, \bar{G}_{i+1\,k_{i+1}}\}$$

is a refinement of the covering $\mathscr{G}_i$ for each i. We can also assume that $\bar{G}_{ij} \cap L_0$ is a subarc of L_0 and it lies in G_{ij} except of its end-points.

Now we shall construct inductively a sequence $L_0, L_1, \ldots$ of arcs joining p with q and containing A and a sequence $C_1, C_2, \ldots$ of arc-region chains with $C_i \supset L_i$, where the arc-region chains are defined similarly as in [13]: A set

$$C = D_0 \cup D_1 \cup \ldots \cup D_{2k-1} \cup D_{2k}$$

is an *arc-region chain* in X if $D_0, D_2, \ldots, D_{2k}$ are some arcs, $D_1, D_3, \ldots, D_{2k-1}$ are compact sets which are the closures of regions in X, $D_i \cap D_j \neq \emptyset$ iff $|i-j| \leqslant 1$, and $D_i \cap D_{i+1}$ is a one-point set which is an end-point of this one of D_i, D_{i+1} which is an arc. After constructing these two sequences, we shall define a desired arc L by the formula

$$L = \bigcap_{i=1}^{\infty} C_i.$$

To define the arc L_1, consider the covering $\mathscr{G}_1$ of A and for each set $\bar{G}_{1j}$ denote by J_{1j} the arc $\bar{G}_{1j} \cap L_0$. Let

$$\dot{J}_{1j} = \{p_{1j}, q_{1j}\},$$

where p_{1j} is chosen so that if a_{1j}, b_{1j} are, respectively, the first and the last points of $A \cap J_{1j}$ in the ordering $<$, then these points lie on the arc J_{1j} successively as follows: $p_{1j}, a_{1j}, b_{1j}, q_{1j}$. We can and do assume that the succession of the regions $G_{11}, \ldots, G_{1k_1}$ coincides with the ordering $<$ of the points $a_{11}, a_{12}, \ldots, a_{1k_1}$. Now, proceeding by induction and using Lemma 3.5 one can easily construct an arc $L_1 \subset X$ and a sequence of its subarcs

$$J'_{11} = p'_{11}q'_{11}, \ldots, J'_{1k_1} = p'_{1k_1}q'_{1k_1}$$

such that

$$\dot{L}_1 = \{p, q\}, \quad J_{1j} \cap A \subset \mathring{J}'_{1j} \subset J'_{1j} \subset J_{1j}$$

and the ordering of these points on the arc L_1 is as follows:

$$p, p'_{11}, a_{11}, b_{11}, q'_{11}, p'_{12}, a_{12}, b_{12}, q'_{12}, \ldots, p'_{1k_1}, a_{1k_1}, b_{1k_1}, q'_{1k_1}, q.$$

To define the arc-region chain C_1, first construct for any arc J'_{1j} a region $G'_{1j} \subset G_{1j}$ such that

$$\bar{G}'_{1j} \cap L_1 = J'_{1j}, \quad \mathring{J}'_{1j} \subset G'_{1j}, \quad \bar{G}'_{1j} \cap \bar{G}'_{1k} = \emptyset \quad \text{for } j \neq k.$$

It is clear that the set $C_1 = L_1 \cup \bar{G}'_{11} \cup \ldots \cup \bar{G}'_{1k_1}$ can be described in a natural way as an arc-region chain.

Now, to define the arc L_2 we replace each of the arcs J'_{1j} for $j \leqslant k_1$ by another arc J''_{1j} with the same end-points and lying in $\bar{G}'_{1j}$. For this purpose we consider the next covering $\mathscr{G}_2$ of A, where we can assume that $\mathscr{G}_2$ is a refinement of the covering

$$\mathscr{G}'_1 = \{G'_{11}, \ldots, G'_{1k_1}\}.$$

Considering a given region G'_{1j} (which we treat as a space), some subarc of the arc J'_{1j} containing $J'_{1j} \cap A$ and contained in G'_{1j}, and the covering of $J'_{1j} \cap A$ by those elements of $\mathscr{G}_2$ which are contained in G'_{1j}, we proceed now to obtain J''_{1j} similarly as before to obtain L_1. After constructing the arcs $J''_{11}, \ldots, J''_{1k_1}$, we define

$$L_2 = (L_1 \backslash \bigcup_{j=1}^{k_1} J'_{1j}) \cup \bigcup_{j=1}^{k_1} J''_{1j}.$$

Replacing the regions $G_{21}, \ldots, G_{2k_2}$ by smaller ones $G'_{21}, \ldots, G'_{2k_2}$ similarly as before, we construct the arc-region chain

$$C_2 = L_2 \cup \bar{G}'_{21} \cup \ldots \cup \bar{G}'_{2k_2}.$$

We construct inductively the arcs $L_1, L_2, \ldots$ and the arc-region chains $C_1, C_2, \ldots$, where $C_i \supset A$ and $C_i \supset C_{i+1}$ for each i. It is clear that the set

$$L = \bigcap_{i=1}^{\infty} C_i$$

is an arc with end-points p, q containing A (cf. [13]) and, moreover, the ordering $<$ on A coincides with the ordering of this set induced by the natural ordering of L from p to q.

3.7. Lemma. *Let X be an arcwise connected continuum containing more than one point and different (topologically) from S^1. If X is arcwise homogeneous, then no region in X is disconnected by an arc.*

Proof. By Proposition 3.3, X is locally connected and has no locally disconnecting points. Assume that there is a region G in X which is disconnected by an arc $K \subset G$. Replacing K by a subarc L such that there is a component C of $G \backslash K$ with $\dot{L} \subset \bar{C}$ and $(K \backslash L) \cap \bar{C} = \emptyset$ if necessary, we can assume that there is a component C of $G \backslash K$ such that $\dot{K} \subset \bar{C}$.

Moreover, we can assume that there are at least two such components of $G \backslash K$. Indeed, assume that C satisfies $\dot{K} \subset \bar{C}$ and let C' be another component of $G \backslash K$. Denote by K' the subarc of K such that

$$\dot{K}' \subset \bar{C}' \quad \text{and} \quad (K \backslash K') \cap \bar{C}' = \emptyset.$$

Then there are at least two components of $G \backslash K'$ containing $\dot{K}'$ in their closures, namely C' and the component of $G \backslash K'$ containing $C \cup (K \backslash K')$. Consequently, since X is arcwise homogeneous, it follows that for any arc $J \subset X$ there is a region $G \supset J$ disconnected by J and such that there are at least two components of $G \backslash J$ containing $\dot{J}$ in their closures.

Now, choose an arc $J_1 = pq$ in X with a given parametric representation from p to q on the interval $[0, 1] \subset E^1$. For any $t \in [0, 1]$ denote by J_t the subarc of J_1 one end-point of which is p and the other is the point q_t of J_1 with the parameter t. Denote by G_t a region in X containing J_t such that there are at least two components of $G_t \backslash J_t$ containing $\dot{J}_t$ in their closures. Then there are $\varepsilon_0 > 0$ and an uncountable subset T of the interval $]0, 1]$ such that for any $t \in T$ the region G_t contains the ball

$$B_t = \{x \in X: \ d(x, J_t) < \varepsilon_0\}.$$

Thus there is a $t_0 \in]0, 1[\cap T$ such that any neighborhood of t_0 in $]0, 1[$ contains uncountably many elements of T (cf. [5], p. 251). Find a region $G \supset J_{t_0}$ and an interval

$$[t_0 - \delta, t_0 + \delta] \subset]0, 1[$$

such that

$$G \subset \{x \in X: \ d(x, J_{t_0}) < \varepsilon_0/2\}, \qquad J_{t_0+\delta} \subset G,$$

$G \cap J_1$ is a ray with end-point p, and for any $t \in T \cap [t_0 - \delta, t_0 + \delta]$ the region G_t contains G. Since for any $t \in T \cap [t_0 - \delta, t_0 + \delta]$ there are at least two components of $G \backslash J_t$ containing $\dot{J}_t$ in their closures, we infer that there is a component C_t of $G \backslash J_t$ such that $\bar{C}_t$ contains q_t, but contains no other point of the subarc $q_t q$ of J_1. Since the interval $[t_0 - \delta, t_0 + \delta]$ contains uncountably many elements of T, the set $G \backslash J_1$ has uncountably many components, which is a contradiction completing the proof of the lemma.

Proof of Theorem 3.2. Let X satisfy the assumptions of Theorem 3.2. By Proposition 3.3 and Lemma 3.7, X satisfies all the assumptions of Lemma 3.6. Consider two Cantor sets $C, C' \subset X$ and a homeomorphism h_0 of C onto C'. By Corollary 3.4 there is an arc $L \subset X$ containing C and such that both the end-points p, q of L belong to C. Consider the linear ordering of C induced by the natural ordering of the arc L from p to q. The homeomorphism h_0 induces an ordering $<$ of C', which conforms to the topology of C'. By Lemma 3.6, there is an arc $L' = p'q' \subset X$ such that $p' = h_0(p)$, $q' = h_0(q)$, $L' \supset C'$ and, moreover, the ordering $<$ of C' coincides with that induced by the natural ordering of the arc L' from p' to q'. Evidently, the homeomorphism h_0 extends to a homeomorphism h_1 of the arc L onto L'. By the strong arcwise homogeneity of X, the homeomorphism h_1 extends to the desired autohomeomorphism h of X.

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