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ON AN INFINITE SYSTEM OF NON-LINEAR SINGULAR INTEGRAL EQUATIONS IN A EUCLIDEAN SPACE

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1. Introduction. Consider in an n-dimensional Euclidean space E_n, $n \geqslant 3$, a system of $p+1$ closed $(n-1)$-dimensional Lapunov surfaces $S_0, S_1, S_2, \ldots, S_p$ $(p \geqslant 0)$ having no common points. The surface S_0 is the boundary of a bounded region Ω_0 containing the surfaces $S_1, S_2, \ldots$ $\ldots, S_p$. Let Ω denote the set of all those points of the region Ω_0 which do not lie on the surfaces $S_1, S_2, \ldots, S_p$. If $p = 0$, then $\Omega = \Omega_0$. The set Ω is the sum $\sum_{i=1}^{q} \Omega_i$ of separable regions $\Omega_1, \Omega_2, \ldots, \Omega_q$, which may be simply-connected or multi-connected regions.

Let $f(y)$ be a complex function integrable in any one of the regions $\Omega_1, \Omega_2, \ldots, \Omega_q$ and $N(x)$ — a complex function defined in each point $x \neq (0, 0, \ldots, 0)$ by the formula

$$N(x) = K(x')|x|^{-n}, \tag{1}$$

where x' denotes the central projection of the point x on the unitary sphere ω, the centre of which is the point $(0, 0, \ldots, 0)$; we then have $x = |x|\cdot x'$. We assume that the function $K(x')$ satisfies on the sphere ω the condition of Hölder,

$$|K(x')-K(y')| \leqslant k_\omega |x'-y'|^{h_\omega}, \qquad 0 < h_\omega \leqslant 1, \tag{2}$$

and, moreover, the condition

$$\int_\omega K(x')\,dx' = 0. \tag{3}$$

After Zygmund [5] and Pogorzelski [2] and [3] we define the singular integral of the function f over the set Ω by

$$\int_\Omega N(x-y)f(y)\,dy \underset{\mathrm{df}}{=} \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} N(x-y)f(y)\,dy, \tag{4}$$

where Ω_ε denotes the set of all points y of the set Ω for which the distance $|x-y|$ is greater than ε.

In connection with the investigation of properties of the multidimensional singular integrals (4), Pogorzelski [2] introduced a certain class of functions the definition of which will now be recalled.

We denote by $\mathfrak{H}_a^h$ the class of all complex functions $f(x)$ defined for $x \in \Omega$, which satisfy the inequality

$$(5) \qquad |x-x_s|^a \cdot |f(x)| \leqslant M_f$$

and the generalized condition of Hölder

$$(6) \qquad |x-x_s|^{a+h} |f(x)-f(y)| \leqslant K_f |x-y|^h,$$

where $|x-y|$ denotes the Euclidean distance of two arbitrary points x and y situated within any of the regions $\Omega_1, \Omega_2, \ldots, \Omega_q$; x_s is the point of one of the surfaces $S_0, S_1, \ldots, S_p$ for which the distance $|x-x_s|$ reaches, for a fixed $x \in \Omega$, a lower limit; we assume that $|x-x_s| \leqslant |y-y_s|$; the parameters a and h are fixed for a given class and satisfy the conditions

$$(7) \qquad 0 \leqslant a < 1, \quad 0 < h < 1, \quad a+h < 1;$$

the positive constants M_f and K_f may depend on f.

Denote by $\mathfrak{H}_a^h(M_f, K_f)$ the subclass of the class $\mathfrak{H}_a^h$ which is obtained by fixing the values of M_f and K_f independently of f.

In the sequel we make use of the following theorems:

THEOREM OF POGORZELSKI [2]. *If a complex function $f(y)$ defined in the set Ω is of class $\mathfrak{H}_a^h(M_f, K_f)$, $a > 0$, then the function $\varphi(x)$ defined in every point $x \in \Omega$ by the singular integral*

$$(8) \qquad \varphi(x) = \int_\Omega N(x-y) f(y)\, dy$$

is of the class $\mathfrak{H}_a^{h'}(C_1 M_f + C_2 K_f, C_1' M_f + C_2' K_f)$, where $h' = \min(h, h_\omega)$ when $h \neq h_\omega$; C_1, C_2, C_1', and C_2' are positive constants independent of f.

THEOREM OF TICHONOV [4]. *If a continuous operation, defined in a linear, metric, locally convex and complete space, transforms a closed, convex and compact set into itself, then there exists at least one fixed point of this operation.*

2. System of integral equations. Consider in the set Ω an infinite system of non-linear singular integral equations

$$(9) \qquad \varphi_\nu(x) = F_\nu\Big\{x, \int_\Omega N_\nu(x-y) R_\nu[y, \varphi_1(y), \varphi_2(y), \ldots]\, dy, \varphi_1(x), \varphi_2(x), \ldots\Big\} \qquad (\nu = 1, 2, \ldots)$$

with unknown functions $\varphi_1(x), \varphi_2(x), \ldots$

We make the following assumptions:

I. Ω is a set defined as above.

II. $F_\nu(x, v, u_1, u_2, \ldots)$, $\nu = 1, 2, \ldots$, are complex functions defined in the domain

$$x \in \Omega, \quad v \in \Pi, \quad u_i \in \Pi, \quad i = 1, 2, \ldots, \tag{10}$$

where Π denotes a plane of a complex variable; moreover,

$$|F_\nu(x, v, u_1, u_2, \ldots)| \leqslant \frac{m_F}{|x - x_s|^a} + k_F\Big(|v| + \sum_{i=1}^{\infty} m_i |u_i|\Big) \tag{11}$$

and

$$|F_\nu(x, v, u_1, u_2, \ldots) - F_\nu(\tilde{x}, \tilde{v}, \tilde{u}_1, \tilde{u}_2, \ldots)| \leqslant \frac{k'_F |x - \tilde{x}|^h}{|x - x_s|^{a+h}} + k_F\Big(|v - \tilde{v}| + \sum_{i=1}^{\infty} k_i |u_i - \tilde{u}_i|\Big), \quad \nu = 1, 2, \ldots, \tag{12}$$

where x and $\tilde{x}$ are arbitrary points situated in any one (both in the same) of the regions $\Omega_1, \Omega_2, \ldots, \Omega_q$; we assume that $|x - x_s| \leqslant |\tilde{x} - \tilde{x}_s|$. Indices a and h satisfy conditions (7); moreover, $a > 0$, $m_i > 0$ and $k_i > 0$ for $i = 1, 2, \ldots$; we assume the convergence of numerical series $\sum_{i=1}^{\infty} m_i$ and $\sum_{i=1}^{\infty} k_i$, and denote their sums by m_∞ and k_∞ respectively; m_F, k_F and k'_F are given positive constants.

III. $N_\nu(x)$, $\nu = 1, 2, \ldots$, are complex functions defined in every point $x \neq (0, 0, \ldots, 0)$ by the formula $N_\nu(x) = K_\nu(x')|x|^{-n}$, where the meaning of x' is the same as in formula (1). Functions $K_\nu(x')$, $\nu = 1, 2, \ldots$, satisfy the condition of Hölder with common constant k_ω and common index exponent $h_\omega > h$; moreover,

$$\int_\omega K_\nu(x')\, dx' = 0, \quad \nu = 1, 2, \ldots \tag{13}$$

IV. $R_\nu(y, w_1, w_2, \ldots)$, $\nu = 1, 2, \ldots$, are complex functions defined in the domain

$$y \in \Omega, \quad w_i \in \Pi, \quad i = 1, 2, \ldots; \tag{14}$$

moreover,

$$|R_\nu(y, w_1, w_2, \ldots)| \leqslant \frac{m_R}{|y - y_s|^a} + \sum_{i=1}^{\infty} m_{Ri} |w_i| \tag{15}$$

and

$$(16)\qquad |R_\nu(y, w_1, w_2, \ldots)-R_\nu(\tilde{y}, \tilde{w}_1, \tilde{w}_2, \ldots)| \leqslant \frac{k_R|y-\tilde{y}|^h}{|y-y_s|^{a+h}}+\sum_{i=1}^{\infty} k_{Ri}|w_i-\tilde{w}_i|,$$

where y and $\tilde{y}$ are arbitrary points situated in one of the regions $\Omega_1, \ldots, \Omega_q$; $|y-y_s| \leqslant |\tilde{y}-\tilde{y}_s|$, $m_{Ri}>0$ and $k_{Ri}>0$ for $i=1,2,\ldots$ We assume the convergence of numerical series $\sum_{i=1}^{\infty} m_{Ri}$ and $\sum_{i=1}^{\infty} k_{Ri}$, and we denote their sums by m_{R_∞} and k_{R_∞} respectively; m_R and k_R are given positive constants.

V. The condition

$$(17)\qquad k_F<\min\left[\frac{1}{k_{R_\infty}(C_2+C_2')+k_\infty}, \frac{1}{m_{R_\infty}(C_1+C_1')+m_\infty}\right]$$

is satisfied, where C_1, C_2, C_1' and C_2' are constants as defined in the theorem of Pogorzelski.

3. THEOREM. *If assumptions* I-V *are satisfied, then the system* (9) *has at least one solution in the class* $\mathfrak{H}_a^h$.

Proof. Consider a space Λ the points U of which are all infinite sequences $\{\varphi_n(x)\}$ of complex functions defined in the set Ω, continuous in every region Ω_j, $j=1,2,\ldots,q$, and satisfying the condition

$$(18)\qquad \sup_{\Omega}[|x-x_s|^{a+h}|\varphi_n(x)|]<\infty.$$

We define the sum of two points $U=\{\varphi_n(x)\}$, $V=\{\tilde{\varphi}_n(x)\}$ of the space Λ and the product of a point and a number by

$$(19)\qquad U+V=\{\varphi_n(x)+\tilde{\varphi}_n(x)\}, \qquad \lambda U=\{\lambda\varphi_n(x)\}.$$

For every point $U\in\Lambda$, we define an infinite sequence of pseudonorms

$$(20)\qquad \|U\|_n=\sup_{\Omega}[|x-x_s|^{a+h}|\varphi_n(x)|].$$

We define the distance $\delta(U, V)$ of points U and V of the space Λ by the formula

$$(21)\qquad \delta(U, V)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|U-V\|_n}{1+\|U-V\|_n}.$$

The space Λ is linear, metric, locally convex and complete.

Let us consider a set $Z(\varrho, \varkappa)$ in the space Λ formed of all those points $\{\varphi_n(x)\}$, for which the conditions

$$(22)\qquad |x-x_s|^a|\varphi_n(x)| \leqslant \varrho, \qquad |x-x_s|^{a+h}|\varphi_n(x)-\varphi_n(y)| \leqslant \varkappa|x-y|^h \qquad (n = 1, 2, \ldots)$$

are satisfied, where ϱ and $\varkappa$ are some positive constants.

The set $Z(\varrho, \varkappa)$ is closed and convex.

Let

$$(23)\qquad \Phi(x) = \begin{cases} |x-x_s|^{a+h}\varphi(x) & \text{for} \quad x \in \Omega, \\ 0 & \text{for} \quad x \in \sum_{k=0}^{p} S_k, \end{cases}$$

where $\varphi(x) \in \mathfrak{H}_a^h(\varrho, \varkappa)$. The so defined functions $\Phi(x)$ are uniformly bounded and uniformly continuous in a bounded domain $\Omega_0 + S_0$. So the set of all of them is compact by virtue of the theorem of Arzelà. Hence it follows that the subset $Z_N(\varrho, \varkappa)$ of the set $Z(\varrho, \varkappa)$ formed of those of its points $\{\varphi_n(x)\}$, for which $\varphi_n(x) \equiv 0$ if $n > N$, is compact as well. Because of the definition (21) of the distance and of the theorem of Fréchet [1] we state that also the set $Z(\varrho, \varkappa)$ is compact.

Referring to equations (9), consider the operation defined for points of $Z(\varrho, \varkappa)$ by the totality of equations

$$(24)\qquad \psi_\nu(x) = F_\nu\Big\{x, \int_\Omega N_\nu(x-y)R_\nu[y, \varphi_1(y), \varphi_2(y), \ldots]dy, \varphi_1(x), \varphi_2(x), \ldots\Big\} \qquad (\nu = 1, 2, \ldots).$$

We shall now show that the constants ϱ and $\varkappa$ may be chosen so that operation (24) would transform the set $Z(\varrho, \varkappa)$ into itself.

Since, on the basis of assumption IV and conditions (22),

$$(25)\qquad |y-y_s|^a|R_\nu[y, \varphi_1(y), \varphi_2(y), \ldots]| \leqslant m_R + \varrho m_{R_\infty}$$

and

$$(26)\qquad |y-y_s|^{a+h}|R_\nu[y, \varphi_1(y), \varphi_2(y), \ldots] - R_\nu[\tilde{y}, \varphi_1(\tilde{y}), \varphi_2(\tilde{y}), \ldots]| \leqslant (k_R + \varkappa k_{R_\infty})|y-\tilde{y}|^h, \qquad \nu = 1, 2, \ldots,$$

where y and $\tilde{y}$ are arbitrary points situated in any of regions $\Omega_1, \Omega_2, \ldots, \Omega_q$ (both in the same), and $|y-y_s| \leqslant |\tilde{y}-\tilde{y}_s|$; thus, on the basis of the above quoted theorem of Pogorzelski, we have

$$(27)\qquad |x-x_s|^a\Big|\int_\Omega N_\nu(x-y)R_\nu[y, \varphi_1(y), \varphi_2(y), \ldots]dy\Big| \leqslant C_1(m_R + \varrho m_{R_\infty}) + C_2(k_R + \varkappa k_{R_\infty})$$

and

$$(28)\qquad |x-x_s|^{a+h}\left|\int_{\Omega}[N_\nu(x-y)-N_\nu(\tilde{x}-y)]R_\nu[y,\varphi_1(y),\varphi_2(y),\ldots]dy\right|$$
$$\leqslant [C_1'(m_R+\varrho m_{R_\infty})+C_2'(k_R+\varkappa k_{R_\infty})]|x-\tilde{x}|^h \qquad (\nu=1,2,\ldots),$$

where x and $\tilde{x}$ are arbitrary points of any of the regions $\Omega_1, \Omega_2, \ldots, \Omega_q$, $|x-x_s|\leqslant|\tilde{x}-\tilde{x}_s|$.

Next, taking under consideration assumption II, we find that

$$(29)\qquad |x-x_s|^a|\psi_\nu(x)|\leqslant m_F+k_F[C_1(m_R+\varrho m_{R_\infty})+C_2(k_R+\varkappa k_{R_\infty})+\varrho m_\infty]$$

and

$$(30)\qquad |x-x_s|^{a+h}|\psi_\nu(x)-\psi_\nu(\tilde{x})|$$
$$\leqslant \{k_F'+k_F[C_1'(m_R+\varrho m_{R_\infty})+C_2'(k_R+\varkappa k_{R_\infty})+\varkappa k_\infty]\}|x-\tilde{x}|^h$$
$$(\nu=1,2,\ldots).$$

Operation (24) transforms thus the set $Z(\varrho,\varkappa)$ into itself if the constants ϱ and $\varkappa$ satisfy the inequalities

$$(31)\qquad \begin{aligned} &m_F+k_F(C_1m_R+C_2k_R)+k_F(C_1m_{R_\infty}+m_\infty)\varrho+k_FC_2k_{R_\infty}\varkappa\leqslant\varrho,\\ &k_F'+k_F(C_1'm_R+C_2'k_R)+k_FC_1'm_{R_\infty}\varrho+k_F(C_2'k_{R_\infty}+k_\infty)\varkappa\leqslant\varkappa. \end{aligned}$$

Simple calculation leads to the conclusion that condition (17) guarantees the existence of a pair of positive numbers $(\varrho_0,\varkappa_0)$ satisfying (31). Therefore, operation (24) transforms $Z(\varrho_0,\varkappa_0)$ into itself.

Next, we shall show that operation (24) is continuous. Let $\{U_j\}$ be an arbitrary sequence of points $U_j=\{\varphi_n^j(x)\}$ of the set $Z(\varrho_0,\varkappa_0)$, convergent to the point $U=\{\varphi_n(x)\}$ of this set in the sense of the metric (21). We shall show that the sequence $\{V_j\}$ of points $V_j=\{\psi_n^j(x)\}$ corresponding to the sequence $\{U_j\}$ in transformation (24) is convergent to the point $V=\{\psi_n(x)\}$ which is the image of the point U in this transformation.

Since $\lim\limits_{j\to\infty}\delta(U_j,U)=0$, we have

$$\lim_{j\to\infty}\|U_j-U\|_n=0 \quad\text{ for }\quad n=1,2,\ldots$$

It suffices to show that

$$\lim_{j\to\infty}\|V_j-V\|_n=0 \quad\text{ for }\quad n=1,2,\ldots$$

Consider the product $|x-x_s|^{\alpha+h}|\psi_n^j(x)-\psi_n(x)|$, which, by virtue of (24) and (12), has the estimate

$$(32)\quad |x-x_s|^{\alpha+h}|\psi_n^j(x)-\psi_n(x)| \leqslant |x-x_s|^{\alpha+h}k_F\Big|\int_\Omega N_n(x-y)\{R_n[y,\varphi_1^j(y),\varphi_2^j(y),\ldots]- -R_n[y,\varphi_1(y),\varphi_2(y),\ldots]\}dy\Big|+k_F\sum_{i=1}^{\infty}k_i\|U_j-U\|_i.$$

We have

$$(33)\qquad \|U_j-U\|_i\leqslant 2\varrho\sup_\Omega|x-x_s|^h$$

for $j=1,2,\ldots$ and $i=1,2,\ldots$ Therefore, because of the convergence of the series $\sum_{i=1}^{\infty}k_i$, the series $\sum_{i=1}^{\infty}k_i\|U_j-U\|_i$ is convergent. Moreover, its sum tends to zero when $j\to\infty$. We may investigate the component appearing at the right-hand side of the estimate (32) so as in the paper [3]; it leads to the conclusion that this component also tends to zero when $j\to\infty$. Consequently,

$$\lim_{j\to\infty}\|V_j-V\|_n=0,$$

which shows the continuity of operation (24).

Since all the assumptions of the above quoted theorem of Tichonov are satisfied, the proof is complete.

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