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ON THE SURJECTIVE SPAN AND SEMISPAN OF CONNECTED METRIC SPACES

BY

A. LELEK (DETROIT, MICHIGAN)

In 1964, I introduced the concept of the span of a metric space (see [4], p. 209). Since that time, several topologists, including H. Cook, W. Holsztyński and D. Zaremba, have asked questions (private communication) which actually dealt with some modified versions of the span, although they were not always explicitly defined or used in those questions. In this note I provide the definitions and try to compare the span with these new quantities (Section 1). An application to the theory of mappings of continua is given (Section 2). The results have been announced (1) earlier in [7].

1. Preliminaries and examples. Let X be a connected metric nonempty space. The standard projections of the product $X\times X$ onto X are denoted by p_1 and p_2: $p_1(x, x') = x$ and $p_2(x, x') = x'$ for $(x, x')\in X\times X$. We define the *surjective span* $\sigma^*(X)$ (respectively, the *surjective semis span* $\sigma_0^*(X)$) of X to be the least upper bound of the set of real number- $a\geqslant 0$ with the following property: there exists a connected set $C_a\subset X\times X$ such that $a\leqslant \operatorname{dist}(x, x')$ for $(x, x')\in C_a$ and $p_1(C_a) = p_2(C_a) = X$ (respectively, $p_1(C_a) = X$). The *span* $\sigma(X)$ and the *semispan* $\sigma_0(X)$ of X are defined by the formulae

$$\sigma(X) = \operatorname{Sup}\{\sigma^*(A)\colon A\subset X,\ A\neq\emptyset \text{ connected}\}, \tag{1}$$

$$\sigma_0(X) = \operatorname{Sup}\{\sigma_0^*(A)\colon A\subset X,\ A\neq\emptyset \text{ connected}\}. \tag{2}$$

It is rather apparent that the above definition of the span, formula (1), is equivalent to that given in [4]. Some other equivalent definitions of the semispan, the surjective span and the surjective semispan of X are also possible, for instance, by means of pairs of continuous mappings

(1) The paper was also presented in a colloquium lecture (May 8, 1975) to the Mathematics Faculty of State University of New York at Buffalo.

of connected spaces into X (cf. [4], p. 209). Proposition 1.1 which follows is a direct consequence of the definitions.

1.1. *Each connected metric non-empty space X satisfies the following inequalities:*

$$0 \leqslant \sigma^*(X) \leqslant \sigma(X) \leqslant \sigma_0(X) \leqslant \operatorname{diam} X, \tag{3}$$

$$0 \leqslant \sigma^*(X) \leqslant \sigma_0^*(X) \leqslant \sigma_0(X) \leqslant \operatorname{diam} X. \tag{4}$$

Each connected non-empty subset A of X satisfies the inequalities

$$\sigma(A) \leqslant \sigma(X) \quad \textit{and} \quad \sigma_0(A) \leqslant \sigma_0(X). \tag{5}$$

Remark. Analogues of inequalities (5) for the surjective span and the surjective semispan do not hold (see 1.4).

1.2. *If X is a bounded connected non-empty subset of the real line, then $\sigma_0^*(X) = 0$.*

Proof. Suppose, on the contrary, that $\sigma_0^*(X) > 0$. Then there exist a number $a_0 > 0$ and a connected set $C_{a_0} \subset X \times X$ such that $a_0 \leqslant |x - x'|$ for $(x, x') \in C_{a_0}$ and $p_1(C_{a_0}) = X$. Also, there exist points $x_1, x_2 \in X$ such that

$$x_1 - a_0 < \operatorname{Inf}\{x:\ x \in X\} \leqslant \operatorname{Sup}\{x;\ x \in X\} < x_2 + a_0.$$

Since $p_1(C_{a_0}) = X$, there are points $x_i' \in X$ such that $(x_i, x_i') \in C_{a_0}$ $(i = 1, 2)$. Hence

$$x_1 - a_0 < x_i' < x_2 + a_0 \qquad (i = 1, 2)$$

and $a_0 \leqslant |x_i - x_i'|$ which implies that $x_1 < x_1'$ and $x_2 > x_2'$. But the set C_{a_0} is connected. Consequently, there exists a pair $(x_0, x_0') \in C_{a_0}$ such that $x_0 = x_0'$. This contradicts the fact that $|x_0 - x_0'| \geqslant a_0 > 0$.

1.3. *If X is a connected non-empty subset of an arc, then $\sigma(X) = \sigma^*(X) = \sigma_0(X) = \sigma_0^*(X) = 0$.*

Proof. Let $X \subset Y$, where Y is an arc, i.e., a set homeomorphic to the unit segment $[0, 1]$ of the real line. By 1.2, we have $\sigma_0^*(A) = 0$ for each connected non-empty set $A \subset [0, 1]$, whence $\sigma_0([0, 1]) = 0$ by (2). It follows from 2.1 that $\sigma_0(Y) = 0$. Thus $\sigma_0(X) = 0$ by (5), as well as $\sigma(X) = \sigma^*(X) = 0$ and $\sigma_0^*(X) = 0$ by (3) and (4), respectively.

1.4. Example. *There exists a connected graph X on the plane such that $\sigma(X) = \sigma_0(X) = 1$ and $\sigma^*(X) = \sigma_0^*(X) = \frac{1}{2}$.*

Proof. Let A denote the circle

$$A = \{(t, u):\ t^2 + u^2 = \tfrac{1}{4}\}$$

and let B denote the segment $[-\frac{1}{2}, 0]$ of the t-axis. We define X to be the union $X = A \cup B$. Setting

$$C_1 = \big\{\big((t, u), (-t, -u)\big):\ (t, u) \in A\big\},$$

we have the connected set $C_1 \subset A \times A$ such that $\operatorname{dist}(x, x') = 1$ for $(x, x') \in C_1$ and $p_1(C_1) = p_2(C_1) = A$. Thus $1 \leqslant \sigma^*(A)$, whence $1 \leqslant \sigma(X)$. Since

$$\sigma(X) \leqslant \sigma_0(X) \leqslant \operatorname{diam} X = 1,$$

by (3), we get $\sigma(X) = \sigma_0(X) = 1$. On the other hand, the set

$$C = C_1 \cup \{(q, x) \colon x \in B\} \cup \{(x, q) \colon x \in B\},$$

where $q = (\frac{1}{2}, 0)$, is connected and $C \subset X \times X$. Moreover, $\frac{1}{2} \leqslant \operatorname{dist}(x, x')$ for $(x, x') \in C$ and $p_1(C) = p_2(C) = X$, so that $\frac{1}{2} \leqslant \sigma^*(X)$. By (4), it remains to show that $\sigma_0^*(X) \leqslant \frac{1}{2}$. Suppose that C' is any subset of the product $X \times X$ with $p_1(C') = X$. The origin $o = (0, 0)$ is a point of X, whence $(o, x') \in C'$ for at least one point $x' \in X$. But then $\operatorname{dist}(o, x') \leqslant \frac{1}{2}$. This implies $\sigma_0^*(X) \leqslant \frac{1}{2}$, according to the definition of the surjective semispan.

Remarks. The graph described in 1.4 is not acyclic: it contains a simple closed curve. Nevertheless, there exists an acyclic graph [8] with the same span properties. The following two problems are suggested by Example 1.4:

PROBLEM 1. Is it true that $\sigma(X) \leqslant 2\sigma^*(X)$ for each connected metric non-empty space X? (**P 994**)

PROBLEM 2. Is it true that $\sigma_0(X) \leqslant 2\sigma_0^*(X)$ for each connected metric non-empty space X? (**P 995**)

An affirmative solution of Problem 1 would imply, by (3), that $\sigma(X) = 0$ if and only if $\sigma^*(X) = 0$. This would answer a question raised by D. Zaremba in a letter to the author. By a *simple n-od*, where $n = 3, 4, \ldots$, we mean any metric space which is the union of n arcs having a common end-point and being pairwise disjoint except at that point. Simple 3-ods are also called *simple triods*. The simple 4-od, described in 1.5 below, had been constructed in [10] for another purpose, and some of its features were investigated in [6].

1.5. Example. *There exists a simple 4-od X in the 3-space such that $\sigma_0^*(X) = \sigma_0(X) = 1$ and $\sigma^*(X) = \sigma(X) = \frac{1}{2}$.*

Proof. We use the ordinary Pythagorean distance for the 3-space R^3. Given two points $x, y \in R^3$, we denote by $\overline{xy}$ the straight-line segment having x any y as the end-points. For $i = 1, 2, 3$, we take the points

$$q_i = \left(\frac{1}{2}\cos\frac{2\pi i}{3}, \frac{1}{2}\sin\frac{2\pi i}{3}, 0\right), \quad r_i = \left(\cos\frac{2\pi(i+1)}{3}, \sin\frac{2\pi(i+1)}{3}, 0\right),$$

and let $o = (0, 0, 0)$ and $r_0 = (0, 0, 1)$. We define X to be the union

$$X = S_0 \cup S_1 \cup S_2 \cup S_3,$$

where $S_0 = \overline{or_0}$ and $S_i = \overline{oq_i} \cup \overline{q_i r_i}$ $(i = 1, 2, 3)$. The origin o is a common end-point of all the four arcs S_i $(i = 0, 1, 2, 3)$ which are pairwise disjoint except at o. Thus X is a simple 4-od in R^3.

The set $C_1 \subset X \times X$ defined by the formula

$$C_1 = [(S_0 \cup S_1) \times \{r_2\}] \cup [(S_0 \cup S_2) \times \{r_3\}] \cup [(S_0 \cup S_3) \times \{r_1\}] \cup \cup [\{r_0\} \times (S_1 \cup S_2 \cup S_3)]$$

is the union of four connected sets the first three of which meet the fourth one. Hence C_1 is a connected set, and we notice that $1 \leqslant \operatorname{dist}(x, x')$ for $(x, x') \in C_1$. Also, we have $p_1(C_1) = X$, so that $1 \leqslant \sigma_0^*(X)$.

To prove that $\sigma_0(X) \leqslant 1$, suppose, on the contrary, that $\sigma_0(X) > 1$. Then there exists, by (2), a connected non-empty set $A \subset X$ such that $\sigma_0^*(A) > 1$. Consequently, there has to be a connected set $C \subset A \times A$ such that $1 < \operatorname{dist}(x, x')$ for $(x, x') \in C$ and $p_1(C) = A$. The origin o is of distance at most 1 to each point of X, whence o cannot belong to $p_1(C)$. We get $o \notin A$, i.e., $A \subset X \setminus \{o\}$. But the point o cuts the simple 4-od X into the four components $S_i \setminus \{o\}$. Since the set A is connected, there exists a subscript $j = 0, 1, 2, 3$ such that $A \subset S_j \setminus \{o\}$, and we obtain $\sigma_0(S_j) > 1$. The set S_j, however, is an arc, and thus $\sigma_0(S_j) = 0$ by 1.3. This is a contradiction showing that $\sigma_0(X) \leqslant 1$. We conclude that $\sigma_0^*(X) = \sigma_0(X) = 1$ by (4).

The set $C' \subset X \times X$ given by the formula

$$C' = [(S_0 \cup S_1 \cup S_2) \times \{r_3\}] \cup [(S_0 \cup S_3) \times \{r_1\}] \cup \cup [\{r_0\} \times (S_1 \cup S_2 \cup S_3)] \cup [\{r_3\} \times (S_0 \cup S_1)]$$

is the union of four connected sets. The first one and the third one of them both contain the point (r_0, r_3), the second and the fourth ones contain (r_3, r_1), while the second and the third ones contain (r_0, r_1). Hence C' is a connected set. Observe that $\frac{1}{2} \leqslant \operatorname{dist}(x, x')$ for $(x, x') \in C'$. We also have $p_1(C') = p_2(C') = X$. It follows that $\frac{1}{2} \leqslant \sigma^*(X)$. On the other hand, it is known that $\sigma(X) = \frac{1}{2}$ (see [6], Example 4.2). By (3), the equalities $\sigma^*(X) = \sigma(X) = \frac{1}{2}$ hold.

Remarks. The following two problems are suggested by Example 1.5:

PROBLEM 3. Is it true that $\sigma_0^*(X) \leqslant 2\sigma^*(X)$ for each connected metric non-empty space X? (**P 996**)

PROBLEM 4. Is it true that $\sigma^*(T) = \sigma_0^*(T)$ for each simple triod T? (**P 997**)

An affirmative solution of Problem 3 would imply, by (1) and (2), that $\sigma_0(X) \leqslant 2\sigma(X)$ for each connected metric non-empty space X. Similarly, an affirmative solution of Problem 4 would imply that $\sigma(T) = \sigma_0(T)$ for each simple triod T. Span properties of a simple triod constructed in [8] also seem to support the conjecture stated in Problem 4.

2. Mappings of continua. By a *continuum* we understand a compact connected metric non-empty space.

2.1. *If X is a connected metric non-empty space, $\tau = \sigma, \sigma^*, \sigma_0, \sigma_0^*$, and $\varepsilon > 0$, then, for each continuous mapping $f\colon X \to Y$ of X onto a metric space Y, there exist points $x_0, x_0' \in X$ such that*

$$\operatorname{dist}(x_0, x_0') > \tau(X) - \varepsilon \quad \textit{and} \quad \operatorname{dist}[f(x_0), f(x_0')] < \tau(Y) + \varepsilon.$$

Hence if, in addition, X is a continuum, then there exist points $x_1, x_1' \in X$ such that

$$\operatorname{dist}(x_1, x_1') \geqslant \tau(X) \quad \textit{and} \quad \operatorname{dist}[f(x_1), f(x_1')] \leqslant \tau(Y).$$

Thus, if X is a continuum and $\tau(Y) = 0$, then there is a point $y \in Y$ such that $\tau(X) \leqslant \operatorname{diam}[f^{-1}(y)]$. As a result, the continua whose span (respectively, surjective span, semispan, and surjective semispan) is zero are invariant under homeomorphisms.

Proof. The standard projections commute with f. More precisely, we have [2]

$$fp_i(C) = p_i(f\times f)(C) \quad (i = 1, 2;\ C \subset X\times X). \tag{6}$$

If $\tau = \sigma$ (respectively, $\tau = \sigma_0$), there exists, by (1) (respectively, by (2)), a connected non-empty set $A \subset X$ such that $\sigma^*(A) > \tau(X) - \varepsilon$ (respectively, $\sigma_0^*(A) > \tau(X) - \varepsilon$). Then there exist a number $a_0 > \tau(X) - \varepsilon$ and a connected set $C_{a_0} \subset A \times A$ such that $a_0 \leqslant \operatorname{dist}(x, x')$ for $(x, x') \in C_{a_0}$ and $p_1(C_{a_0}) = p_2(C_{a_0}) = A$ (respectively, $p_1(C_{a_0}) = A$). Setting

$$B = f(A) \quad \text{and} \quad D = (f\times f)(C_{a_0}),$$

we get a connected non-empty set $B \subset Y$ and a connected set $D \subset B \times B$. It follows from (6) that $p_1(D) = p_2(D) = B$ (respectively, $p_1(D) = B$). On the other hand, $\sigma^*(B) < \tau(Y) + \varepsilon$ by (1) (respectively, $\sigma_0^*(B) < \tau(Y) + \varepsilon$ by (2)), whence it is not true that $\tau(Y) + \varepsilon \leqslant \operatorname{dist}(y, y')$ for $(y, y') \in D$. In other words, there exists a pair $(y_0, y_0') \in D$ such that $\operatorname{dist}(y_0, y_0') < \tau(Y) + \varepsilon$. Let $(x_0, x_0') \in C_{a_0}$ be a pair such that $(y_0, y_0') = (f\times f)(x_0, x_0')$. Clearly, the points x_0 and x_0' satisfy conditions required in 2.1.

If $\tau = \sigma^*$ (respectively, $\tau = \sigma_0^*$), then $\sigma^*(X) > \tau(X) - \varepsilon$ (respectively, $\sigma_0^*(X) > \tau(X) - \varepsilon$). The remainder of the proof is a replica of the above argument for $\tau = \sigma$ (respectively, for $\tau = \sigma_0$) with A and B replaced by X and Y, respectively.

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(2) Here we use the same symbol p_i to denote the corresponding projection of $X \times X$ onto X and of $Y \times Y$ onto Y. The mapping $f\times f\colon X \times X \to Y \times Y$ is defined by $(f\times f)(x, x') = (f(x), f(x'))$ for $(x, x') \in X \times X$.

Remarks. Proposition 2.1 generalizes an earlier result (see [4], p. 209-210) which had covered only the case of $\tau = \sigma$. Even in this case a much stronger conclusion is possible provided the space X is assumed to be a continuum of a special type. Our next result will explore such a possibility (see 2.4). Before proving it, we recall the definitions of some classes of continua.

By a *tree* we understand a continuum that is an acyclic graph, i.e., a set homeomorphic to a one-dimensional connected polyhedron which contains no simple closed curve. A continuum X is *unicoherent* if the intersection of any two continua whose union is X is a continuum. Each simple n-od ($n = 3, 4, \ldots$) is a tree, and each tree is a unicoherent continuum. We say that a continuum X is *arc-like* (respectively, *tree-like*, and *P-unicoherent*) provided, for each $\varepsilon > 0$, there exists a finite open cover of X whose elements have diameters less than ε and whose nerve (see [2], p. 318), if non-degenerate, is an arc (respectively, a tree, and a unicoherent connected polyhedron). Thus each arc-like continuum is tree-like, and each tree-like continuum is P-unicoherent. The solenoid is an example of a unicoherent continuum that is not P-unicoherent. It is not difficult to check, however, that each P-unicoherent continuum is unicoherent.

2.2. LEMMA. *If X is a P-unicoherent continuum and $0 \leqslant \beta \leqslant \sigma^*(X)$ (respectively, $0 \leqslant \beta \leqslant \sigma_0^*(X)$), then there exists a continuum $K_\beta \subset X \times X$ such that* $\operatorname{dist}(x, x') = \beta$ *for $(x, x') \in K_\beta$ and $p_1(K_\beta) = p_2(K_\beta) = X$ (respectively, $p_1(K_\beta) = X$).*

Proof. Without loss of generality, we can assume that X is a subset of the Hilbert space R^ω. Moreover, there exists a metric for R^ω which is an extension of the given metric in X (see [1], p. 353). We use R^ω equipped with this distance. Since X is a P-unicoherent continuum, it has finite open covers whose elements have arbitrarily small diameters and whose nerves are unicoherent connected polyhedra. It follows that, for each $n = 1, 2, \ldots$, there exist a unicoherent connected polyhedron $P_n \subset R^\omega$ contained in the $(1/n)$-neighbourhood of X in R^ω and a continuous mapping $f_n\colon X \to P_n$ such that f_n is a $(1/n)$-translation, i.e., $\operatorname{dist}[x, f_n(x)] < 1/n$ for $x \in X$ (see [2], p. 319, 324 and 330). Then the set $P_n \times P_n$ is contained in the $(\sqrt{2}/n)$-neighbourhood of $X \times X$ in $R^\omega \times R^\omega$, and $f_n \times f_n$ is a $(\sqrt{2}/n)$-translation transforming $X \times X$ into $P_n \times P_n$. Since $X \times X$ and $P_n \times P_n$ ($n = 1, 2, \ldots$) are compact sets in $R^\omega \times R^\omega$, their union

$$Z = (X \times X) \cup \bigcup_{n=1}^{\infty} (P_n \times P_n) \tag{7}$$

is also a compact set. We have $(f_n \times f_n)(X \times X) \subset Z$ for $n = 1, 2, \ldots$

Case 1. $0 \leqslant \beta \leqslant \sigma^*(X)$. If $\sigma^*(X) = 0$, then $\beta = 0$ and the conclusion of Lemma 2.2 is trivial, as one can take K_0 to be the diagonal of the product $X \times X$. Let us assume that $\sigma^*(X) > 0$ and select an infinite sequence $\beta_1, \beta_2, \ldots$ of numbers such that

$$(8) \qquad \beta = \lim_{j\to\infty} \beta_j \quad \text{and} \quad 0 < \beta_j < \sigma^*(X) \quad (j = 1, 2, \ldots).$$

Consequently, there exist numbers $a_j > \beta_j$ and connected sets $C_{a_j} \subset X \times X$ such that $a_j \leqslant \mathrm{dist}(x, x')$ for $(x, x') \in C_{a_j}$ and $p_i(C_{a_j}) = X$ $(i = 1, 2;\ j = 1, 2, \ldots)$. Let $n_j \geqslant j$ be an integer such that $n_j \geqslant 2/(a_j - \beta_j)$ $(j = 1, 2, \ldots)$. The set

$$E_j = (f_{n_j} \times f_{n_j})(C_{a_j}) \quad (j = 1, 2, \ldots)$$

is a connected non-empty subset of $P_{n_j} \times P_{n_j}$. If $(x, x') \in C_{a_j}$, then

$$a_j \leqslant \mathrm{dist}(x, x') \leqslant \mathrm{dist}[x, f_{n_j}(x)] + \mathrm{dist}[f_{n_j}(x), f_{n_j}(x')] + \mathrm{dist}[f_{n_j}(x'), x']$$
$$< \mathrm{dist}[f_{n_j}(x), f_{n_j}(x')] + \frac{2}{n_j} \leqslant \mathrm{dist}[f_{n_j}(x), f_{n_j}(x')] + (a_j - \beta_j),$$

whence $\beta_j < \mathrm{dist}(y, y')$ for $(y, y') \in E_j$ $(j = 1, 2, \ldots)$. The closed set

$$(9) \qquad F_j = \{(y, y') \in P_{n_j} \times P_{n_j} \colon \mathrm{dist}(y, y') = \beta_j\}$$

cuts the product $P_{n_j} \times P_{n_j}$ into the two sets

$$G_j = \{(y, y') \in P_{n_j} \times P_{n_j} \colon \mathrm{dist}(y, y') < \beta_j\},$$
$$H_j = \{(y, y') \in P_{n_j} \times P_{n_j} \colon \mathrm{dist}(y, y') > \beta_j\},$$

and we have $E_j \subset H_j$ $(j = 1, 2, \ldots)$. On the other hand, the diagonal D_j of $P_{n_j} \times P_{n_j}$ is a subset of G_j by (8). Let $d_j \in D_j$ and $e_j \in E_j$ be points. Since F_j separates d_j and e_j in the continuum $P_{n_j} \times P_{n_j}$ which is unicoherent and locally connected, there exists a continuum $L_j \subset F_j$ separating d_j and e_j in $P_{n_j} \times P_{n_j}$ (see [3], p. 434 and 438). If $y \in p_i(E_j)$ $(i = 1, 2;$ $j = 1, 2, \ldots)$, where $p_i\colon R^\omega \times R^\omega \to R^\omega$ denotes the standard projection, then the set $(P_{n_j} \times P_{n_j}) \cap p_i^{-1}(y)$ is a copy of P_{n_j} and it meets both the sets D_j and E_j. All these three sets are connected, so that their union

$$U = D_j \cup E_j \cup [(P_{n_j} \times P_{n_j}) \cap p_i^{-1}(y)]$$

is also a connected set. We notice that $d_j, e_j \in U \subset P_{n_j} \times P_{n_j}$, whence $L_j \cap U \neq \emptyset$. But since

$$L_j \cap (D_j \cup E_j) \subset F_j \cap (G_j \cup H_j) = \emptyset,$$

we get $L_j \cap p_i^{-1}(y) \neq \emptyset$, that is, $y \in p_i(L_j)$. Thus

$$(10) \qquad p_i(E_j) \subset p_i(L_j) \quad (i = 1, 2;\ j = 1, 2, \ldots).$$

The inclusions $L_j \subset F_j$ imply, by (7) and (9), that the continua L_j are subsets of the compact set Z. Then there exists a convergent subsequence $L_{j_1}, L_{j_2}, \ldots$ (where $j_1 < j_2 < \ldots$) whose limit is a continuum (see [3], p. 45, 49 and 139). We define K_β to be this limit, i.e.,

$$K_\beta = \operatorname*{Lim}_{k\to\infty} L_{j_k}.$$

Since

$$L_{j_k} \subset F_{j_k} \subset P_{n_{j_k}} \times P_{n_{j_k}} \qquad (k = 1, 2, \ldots)$$

and the latter set is contained in the $(\sqrt{2}/n_{j_k})$-neighbourhood of $X \times X$ in $R^\omega \times R^\omega$, we conclude that $K_\beta \subset X \times X$. If $(x, x') \in K_\beta$, there exist two infinite sequences of points $y_k, y_k' \in R^\omega$ such that

$$(x, x') = \lim_{k\to\infty}(y_k, y_k') \quad \text{and} \quad (y_k, y_k') \in L_{j_k} \subset F_{j_k},$$

whence $\operatorname{dist}(y_k, y_k') = \beta_{j_k}$ $(k = 1, 2, \ldots)$ by (9). It follows that

$$\operatorname{dist}(x, x') = \lim_{k\to\infty} \operatorname{dist}(y_k, y_k') = \lim_{k\to\infty} \beta_{j_k} = \beta$$

by (8). Finally, we observe that the convergence of L_{j_k} to K_β takes place in the compact set Z, which implies that the projections $p_i(K_\beta)$ are limits of the projections $p_i(L_{j_k})$, respectively. Also, by (6) and (10), we have

$$f_{n_j}(X) = f_{n_j} p_i(C_{a_j}) = p_i(f_{n_j} \times f_{n_j})(C_{a_j}) = p_i(E_j) \subset p_i(L_j)$$
$$(i = 1, 2;\ j = 1, 2, \ldots).$$

The mapping f_n is a $(1/n)$-translation; therefore

$$X = \operatorname*{Lim}_{k\to\infty} f_{n_{j_k}}(X) \subset \operatorname*{Lim}_{k\to\infty} p_i(L_{j_k}) = p_i(K_\beta) \qquad (i = 1, 2),$$

whence $p_1(K_\beta) = p_2(K_\beta) = X$, as K_β is a subset of $X \times X$.

Case 2. $0 \leqslant \beta \leqslant \sigma_0^*(X)$. The proof is similar to that of Case 1, the only difference being that now we have $i = 1$ instead of $i = 1, 2$.

2.3. Lemma. *If X is a tree-like continuum and $0 \leqslant \beta \leqslant \sigma(X)$ (respectively, $0 \leqslant \beta \leqslant \sigma_0(X)$), then there exists a continuum $K_\beta \subset X \times X$ such that $\operatorname{dist}(x, x') = \beta$ for $(x, x') \in K_\beta$ and $p_1(K_\beta) = p_2(K_\beta)$ (respectively, $p_1(K_\beta) \supset p_2(K_\beta)$).*

Proof. Since each non-degenerate subcontinuum of a tree is a tree itself, each subcontinuum of a tree-like continuum is tree-like. Hence each subcontinuum of a tree-like continuum is P-unicoherent.

If $\sigma(X) = 0$ (respectively, $\sigma_0(X) = 0$), then $\beta = 0$ and the conclusion of Lemma 2.3 states a trivial fact, as one can take K_0 to be any subcontinuum of the diagonal of the product $X \times X$. Let us assume that $\sigma(X) > 0$ (respectively, $\sigma_0(X) > 0$) and let $0 \leqslant \beta < \sigma(X)$ (respectively, $0 \leqslant \beta < \sigma_0(X)$). By (1) (respectively, by (2)), there exists a connected non-empty set $A \subset X$ such that $\sigma^*(A) > \beta$ (respectively, $\sigma_0^*(A) > \beta$). Then there exist a number $a_0 \geqslant \beta$ and a connected set $C_{a_0} \subset A \times A$ such that

$a_0 \leqslant \operatorname{dist}(x, x')$ for $(x, x') \in C_{a_0}$ and $p_1(C_{a_0}) = p_2(C_{a_0}) = A$ (respectively, $p_1(C_{a_0}) = A$). The closures C' and X' of C_{a_0} and A in $X \times X$ and X, respectively, are continua and $C' \subset X' \times X'$. Moreover, $a_0 \leqslant \operatorname{dist}(x, x')$ for $(x, x') \in C'$ and $p_1(C') = p_2(C') = X'$ (respectively, $p_1(C') = X'$). Hence $\sigma^*(X') \geqslant a_0$ (respectively, $\sigma_0^*(X') \geqslant a_0$), and X' is a P-unicoherent continuum. By 2.2, there exists a continuum $K_\beta \subset X' \times X' \subset X \times X$ such that $\operatorname{dist}(x, x') = \beta$ for $(x, x') \in K_\beta$ and $p_1(K_\beta) = p_2(K_\beta) = X'$ (respectively, $p_1(K_\beta) = X' \supset p_2(K_\beta)$). The remaining case of $\beta = \sigma(X) > 0$ (respectively, $\beta = \sigma_0(X) > 0$) can be handled by choosing an infinite sequence of positive numbers $\beta_i < \beta$ which converge to β, and then selecting a convergent subsequence of the sequence of corresponding continua $K_{\beta_i} \subset X \times X$. This is because conditions involving distances and projections, and expressed in terms of some equalities and inclusions, are preserved in the process of forming a limit.

Remarks. Continuity properties of the real-valued functions σ, σ^*, σ_0 and σ_0^* have implicitly been utilized in the proof of Lemmas 2.2 and 2.3. For the span σ, they are discussed in [6], and analogous properties are enjoyed by the other three versions of the span: σ^*, σ_0 and σ_0^*. They can easily be formulated and proved using methods developed in [6].

2.4. Theorem. *If X is a P-unicoherent continuum, $\tau = \sigma^*$, σ_0^*, and $0 \leqslant \beta \leqslant \tau(X)$, then, for each continuous mapping $f\colon X \to Y$ of X onto a continuum Y, there exist points $x_0, x_0' \in X$ such that*

$$\operatorname{dist}(x_0, x_0') = \beta \quad \textit{and} \quad \operatorname{dist}[f(x_0), f(x_0')] \leqslant \tau(Y).$$

Moreover, if X is a tree-like continuum, the same conclusion also holds for $\tau = \sigma, \sigma_0$.

Proof. By 2.2 and 2.3, there exists a continuum $K_\beta \subset X \times X$ such that $\operatorname{dist}(x, x') = \beta$ for $(x, x') \in K_\beta$ and one of the following four conditions holds:

$$\begin{aligned} &p_1(K_\beta) = p_2(K_\beta) && (\tau = \sigma),\\ &p_1(K_\beta) = p_2(K_\beta) = X && (\tau = \sigma^*),\\ &p_1(K_\beta) \supset p_2(K_\beta) && (\tau = \sigma_0),\\ &p_1(K_\beta) = X && (\tau = \sigma_0^*). \end{aligned}$$

The set $D = (f \times f)(K_\beta)$ is a subcontinuum of $Y \times Y$. It follows from (6) that one of the following conditions holds:

$$\begin{aligned} &p_1(D) = p_2(D) && (\tau = \sigma),\\ &p_1(D) = p_2(D) = Y && (\tau = \sigma^*),\\ &p_1(D) \supset p_2(D) && (\tau = \sigma_0),\\ &p_1(D) = Y && (\tau = \sigma_0^*). \end{aligned}$$

If $\operatorname{dist}(y, y') > \tau(Y)$ for $(y, y') \in D$, the compactness of D would imply the existence of a number $\varepsilon_0 > 0$ such that $\operatorname{dist}(y, y') \geqslant \tau(Y) + \varepsilon_0$ for $(y, y') \in D$. This, on the other hand, would imply the inequality $\tau(Y) \geqslant \tau(Y) + \varepsilon_0$, a contradiction. Hence there exists a pair $(y_0, y_0') \in D$ such that $\operatorname{dist}(y_0, y_0') \leqslant \tau(Y)$. Let $(x_0, x_0') \in K_\beta$ be a pair such that $(y_0, y_0') = (f \times f)(x_0, x_0')$. Clearly, the points x_0 and x_0' satisfy conditions required in Theorem 2.4.

We write

$$\Delta_f = \{\operatorname{dist}(x, x') \colon x, x' \in X,\ f(x) = f(x')\}.$$

2.5. COROLLARY. *If X is a P-unicoherent continuum, $\tau = \sigma^*, \sigma_0^*$, and $0 \leqslant \beta \leqslant \tau(X)$, then, for each continuous mapping $f\colon X \to Y$ of X onto a continuum Y with $\tau(Y) = 0$, there exist points $x, x' \in X$ such that*

$$\operatorname{dist}(x, x') = \beta \quad \textit{and} \quad f(x) = f(x').$$

Consequently, we have $[0, \tau(X)] \subset \Delta_f$.

Moreover, if X is a tree-like continuum, the same conclusion also holds for $\tau = \sigma, \sigma_0$.

Remarks. For tree-like continua, the inclusion $[0, \sigma_0(X)] \subset \Delta_f$ from 2.5 is stronger than the result of [9] which involves the span $\sigma(X)$. Indeed, there exist, by 1.5, trees X such that $\sigma(X) < \sigma_0(X)$. For P-unicoherent continua, there is an analogue of 2.5 involving $\sigma(X)$ rather than $\sigma^*(X)$ or $\sigma_0^*(X)$ (see [6], Theorem 5.3), but under the assumption that the mapping transforms X into an arc-like continuum. Arc-like continua admit continuous mappings into arcs with arbitrarily small diameters of point-inverses. It follows from 1.3 and 2.1 that each arc-like continuum has span, as well as surjective span, semispan and surjective semispan, all equal to zero. It is still unknown, however, whether or not each continuum of span zero (or of semispan zero) is arc-like (see [5], p. 93).

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WAYNE STATE UNIVERSITY

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