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ON COHOMOLOGICAL DIMENSION OF A SPACE AND ITS STONE–ČECH COMPACTIFICATION

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Let us recall that, for a given family of supports φ on a topological space X, the least integer n for which the Grothendieck cohomology groups $H_\varphi^k(X; \mathscr{A})$ are equal to 0 for every sheaf $\mathscr{A}$ of L-modules and for each $k > n$ is called the *φ-dimension* of X; it is denoted by $\dim_{\varphi,L}(X)$. If φ is a paracompactifying family such that

$$\bigcup \{K \mid K \in \varphi\} = X,$$

then $\dim_{\varphi,L}(X)$ is independent of φ, and in that case $\dim_{\varphi,L}(X)$ is called the *cohomological dimension* of X over the ring L, and is denoted by $\dim_L(X)$. A lot of work has already been done on this dimension function ([3], [1], [4]) and it has proved to be quite useful in the cohomological theory of topological transformation groups ([2], [6]). The relationship between $\dim_L(X)$ and $\dim_L(\beta X)$, where βX is the Stone–Čech compactification of X, however, still remains to be completely investigated. For example, if X is paracompact, it is not known whether or not

$$\dim_L(X) = \dim_L(\beta X) \tag{A}$$

holds for a given ring L. What is known in this direction is the following: For a given sheaf $\mathscr{A}$ of L-modules, let $\dim_{\mathscr{A}}(X)$ be the least integer n such that $H^k(X; \mathscr{A}_U) = 0$ for each $k > n$ and for each $U \subseteq X$ open in X. Then it is easy to see that, for a locally paracompact space X,

$$\dim_L(X) = \sup_{\mathscr{A}} \{\dim_{\mathscr{A}}(X)\}.$$

Then it follows from [7], p. 40, that if X is locally compact and paracompact and G is a finitely generated abelian group, then (treating G as the constant sheaf of Z-modules) $\dim_G(X) = \dim_G(\beta X)$. Even the most easy-looking inequality

$$\dim_L(X) \leqslant \dim_L(\beta X) \tag{B}$$

is not fully established for an arbitrary paracompact space X. Since $\dim_L(X)$ is monotonic for locally closed subsets of X, and because X is locally closed in βX (in fact, open) if and only if X is locally compact, (B) is trivially valid for any locally compact space X. In this paper we prove, however, that, for a locally compact space X, (B) is best possible by showing that for such spaces (A) is, in general, not true. In fact, our main result is

THEOREM 1. *For any given pair* (m, n) *of integers such that* $0 \leqslant m < n \leqslant \infty$ *there exists a locally compact Hausdorff space* $X_{(m,n)}$ *for which*

$$\dim_{\mathbf{Z}}(X_{(m,n)}) = m \quad \text{and} \quad \dim_{\mathbf{Z}}(\beta X_{(m,n)}) = n.$$

Let N be any countable infinite set (e.g., natural numbers). Hence onwards, by an *almost-disjoint collection* $\mathscr{R}$ in N we mean the set

$$\mathscr{R} = \{\lambda \mid \lambda \text{ is an infinite subset of } N \text{ such that,}$$
$$\text{for any two distinct } \lambda, \lambda', \lambda \cap \lambda' \text{ is a finite set}\}.$$

Consider the set $N \cup \mathscr{R}$ and give the topology on this as follows: points of N are isolated and a basic neighbourhood of $\lambda \in \mathscr{R}$ is $\{\lambda\} \cup (\lambda \backslash F)$, where F is some finite subset of λ ([5], Exercise 5 I). Then this space is well known to have the following properties: it is locally compact Hausdorff but not necessarily normal; each point of the space has a neighbourhood base of clopen sets, i.e., $\operatorname{ind}(N \cup \mathscr{R}) = 0$; N is dense and $\mathscr{R}$ is a relatively discrete closed subset of $N \cup \mathscr{R}$; $\mathscr{R}$ is a maximal almost-disjoint collection in N if and only if $N \cup \mathscr{R}$ is pseudocompact. If $\mathscr{R}$ is infinite, then $N \cup \mathscr{R}$ cannot be countably compact. Thus, if $\mathscr{R}$ is infinite and also maximal, then $N \cup \mathscr{R}$ cannot be normal. It is also clear that each finite subset of $N \cup \mathscr{R}$ is a zero-set for any $\mathscr{R}$, and if $\mathscr{R}$ is maximal almost-disjoint, then a subset of $\mathscr{R}$ is a zero-set in $N \cup \mathscr{R}$ implies either it is finite or uncountable.

First of all, we prove

PROPOSITION 1. *Let X be a locally compact Hausdorff space. Then the small inductive dimension* ($\operatorname{ind} X$) *is zero if and only if* $\dim_L(X) = 0$ *for a non-zero ring L.*

Proof. To prove the necessity, first notice that the family c of all compact subsets of X is a paracompactifying family of supports on X with extent X. It suffices to show that every sheaf $\mathscr{A}$ of L-modules on X is c-soft. Let K be a compact subset of X and $s\colon K \to \mathscr{A}$ be a section of $\mathscr{A}$ on K. Now, since K is compact, there is an extension s' of s to some neighbourhood U of K. But $\operatorname{ind} X = 0$ implies that any neighbourhood of K contains a clopen neighbourhood. Let V be a neighbourhood of K which is clopen such that $K \subseteq V \subseteq U$. Then $s' \mid V$ is a section of $\mathscr{A}$ which can be obviously extended (by zero) to X. Hence s has an extension to X. Thus $\mathscr{A}$ is c-soft, and so $\dim_L(X) = 0$ for any non-zero ring L.

To prove the sufficiency, suppose that $\dim_L(X) = 0$ for a non-zero ring L. The abelian group $Z_2 = \{0, 1\}$ can always be regarded as an L-module (possibly trivial). Let us regard Z_2 as the constant sheaf of L-modules on X. Let $x \in X$ be any point and U be an open neighbourhood of x. We can assume that X is compact. Therefore, $X \setminus U$ is compact. We define a section

$$s\colon \{x\} \cup (X \setminus U) \to Z_2$$

by $s(x) = 1$ and $s|(X \setminus U) = 0$. Since $\dim_L(X) = 0$, the constant sheaf Z_2 is c-soft. Now, $\{x\} \cup (X \setminus U)$ being compact, there exists an extension $s'\colon X \to Z_2$ of s. Put

$$V = \{x \in X|\ s'(x) = 1\}.$$

Then V is a clopen set contained in U. Hence $\operatorname{ind} X = 0$.

Now, the following corollary is obvious:

COROLLARY 1. *For the space $X = N \cup \mathscr{R}$, where $\mathscr{R}$ is any almost-disjoint collection in N,* $\dim_Z(X) = 0$.

If X is locally compact, then X is open in βX. Then, it follows from the complementary sum theorem for $\dim_L$ proved in [4] (Corollary 4.10) that

$$\dim_L(\beta X) = \max\{\dim_L(X), \dim_L(\beta X \setminus X)\}.$$

This implies the next

PROPOSITION 2. *If X is a locally compact Hausdorff space such that* $\dim_L(X) = 0$, *then*

$$\dim_L(\beta X) = \dim_L(\beta X \setminus X).$$

This gives then

COROLLARY 2. *For any almost-disjoint collection $\mathscr{R}$ in N,*

$$\dim_Z(\beta(N \cup \mathscr{R})) = \dim_Z[\beta(N \cup \mathscr{R}) \setminus (N \cup \mathscr{R})].$$

The proof of the following proposition can be found in [9] (Theorem 2.1, p. 96).

PROPOSITION 3. *For any given compact metric space X without isolated points, there exist a countably infinite subset N of X and a maximal almost-disjoint collection $\mathscr{R}(X)$ in N such that X is homeomorphic to* $\beta(N \cup \mathscr{R}(X)) \setminus (N \cup \mathscr{R}(X))$.

Since each n-cube I^n ($n = 1, 2, \ldots, \infty$) is a compact metric space without isolated points, by the above proposition there exists a maximal almost-disjoint collection $\mathscr{R}_n$ in the set N of natural numbers such that

$$I^n \approx [\beta(N \cup \mathscr{R}_n) \setminus (N \cup \mathscr{R}_n)].$$

But $\dim_Z(I^n) = n$, which follows easily from the product theorem for $\dim_Z$

and the fact that $\dim_Z(I) = 1$ ([1], pp. 143 and 144). Therefore, we have

COROLLARY 3. *For any $n = 1, 2, \ldots, \infty$, there exists a maximal almost-disjoint collection $\mathscr{R}_n$ in the set N of natural numbers such that*

$$\dim_Z[\beta(N \cup \mathscr{R}_n)\backslash(N \cup \mathscr{R}_n)] = n.$$

Proof of Theorem 1. Let S^m $(m = 1, 2, \ldots)$ be the standard m-sphere and $\mathscr{R}_n$ be the maximal almost-disjoint collection in N of Corollary 3. Define

$$X_{(m,n)} = S^m + (N \cup \mathscr{R}_n)$$

to be the disjoint topological sum of S^m and the space $N \cup \mathscr{R}_n$. Then $X_{(m,n)}$ is the required locally compact Hausdorff space. In fact, using the complementary sum theorem for $\dim_L$ and Corollary 1, we have

$$\dim_Z(X_{(m,n)}) = \max\{\dim_Z(S^m), \dim_Z(N \cup \mathscr{R}_n)\} = \dim_Z(S^m) = m,$$

since S^m is the one-point compactification of $\boldsymbol{R}^m$, and therefore, by the complementary sum theorem for $\dim_L$, $\dim_Z(S^m) = \dim_Z(\boldsymbol{R}^m) = m$ ([1], p. 144).

Next, using the fact that the Stone–Čech compactification distributes over the union $S \cup (X\backslash S)$ if S is clopen in X ([5], p. 90), we have

$$\begin{aligned}\dim_Z(\beta X_{(m,n)}) &= \max\{\dim_Z(S^m), \dim_Z(\beta(N \cup \mathscr{R}_n))\} \\ &= \max\{m, \dim_Z[\beta(N \cup \mathscr{R}_n)\backslash(N \cup \mathscr{R}_n)]\}\end{aligned}$$

by Corollary 2. Consequently, by Corollary 3,

$$\dim_Z(\beta X_{(m,n)}) = n.$$

This completes the proof.

Concluding remarks. (i) We wish to point out that the proof of Proposition 3 makes an essential use of the axiom of choice, and therefore the construction of our example of the spaces $X_{(m,n)}$ also depends on the axiom of choice. It would be desirable to find examples of spaces $X_{(m,n)}$ without using the axiom of choice.

(ii) It is straightforward to observe that in the statement of Theorem 1 there is nothing special about the ground ring Z being the ring of integers. In fact, in view of Proposition 1, Corollaries 1–3 remain valid for any ring L. Moreover, it is immediate from [1], p. 144, that, for a non-zero principal ideal domain L, $\dim_L(I^n) = n$ and $\dim_L(S^m) = m$. Consequently, Theorem 1 remains valid for any non-zero principal ideal domain L.

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