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SOME ASPECTS OF HARMONIC ANALYSIS ON FREE GROUPS

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For a non-commutative free group F we consider the representation $F \ni x \to \Delta(x) \in B(l^2(F))$ defined by

$$[\Delta(x)f](y) = f(x^{-1}yx) = [\varrho(x)\lambda(x)f](y),$$

where ϱ and λ are the right and the left regular representations of F, respectively. We lift Δ to $l^1(F)$ and denote by $C^*_\Delta(F)$ the C^*-algebra obtained by completing $l^1(G)$ in the norm $\|f\|_\Delta = \|\Delta(f)\|$.

The aim of this note is to describe the structure of $C^*_\Delta(F)$ (Section 2, Theorem 1). As a consequence, we obtain some information concerning $B_\Delta(F)$.

1. Preliminaries. We recall some notion and theorems which are basic for everything what follows.

Let G be a locally compact group and let $\Sigma = \Sigma(G)$ denote the space of all *-representations of $L^1(G)$. We fix a subset S of Σ as in [7] and we write

$$N_S = \{f \in L^1(G)\colon \pi(f) = 0 \text{ for } \pi \in S\},$$

$$N'_S = \{f \in C^*(G)\colon \pi(f) = 0 \text{ for } \pi \in S\}.$$

Let $C^*_S(G)$ be the completion of the quotient algebra $L^1(G)/N_S$ with respect to the norm

$$\|f\|_S = \sup\{\|\pi(f)\|\colon \pi \in S\}.$$

PROPOSITION A (Eymard [7]). *The canonical map $f \to \dot{f}$ from $L^1(G)$ onto $L^1(G)/N_S$ extends to a *-homomorphism*

$$a_S\colon C^*(G) \to C^*_S(G)$$

such that $\ker a_S = N'_S$ *and $C^*_S(G)$ is isometrically isomorphic to $C^*(G)/N'_S$.*

We denote by $P_S(G)$ the space of positive-definite functions associated with representations in S, i.e., $p \in P_S(G)$, iff $p(x) = (\pi(x)\xi \mid \xi)$ for a $\pi \in S$

and a $\xi \in H(\pi)$. Let $B_S(G)$ denote the linear span of $P_S(G)$. Now, $B_S(G)$ equipped with a norm turns out (cf. [7]) to be isometrically isomorphic to the dual of $C_S^*(G)$.

For $S_1, S_2 \subset \Sigma$, S_1 weakly contained in S_2, we write $S_1 \prec^{\mathrm{w}} S_2$ if one of the following equivalent statements holds:

(a) $N'_{S_2} \subset N'_{S_1}$.

(b) $B_{S_1}(G) \subset B_{S_2}(G)$.

(c) For every $f \in L^1(G)$, $\|f\|_{S_1} \leqslant \|f\|_{S_2}$.

If $S_1 \prec^{\mathrm{w}} S_2$ and $S_2 \prec^{\mathrm{w}} S_1$, then S_1 is *weakly equivalent* to S_2.

As has been observed by Godement ([9], see also [4]),

$$B_c(G) = \{u \in B(G) \colon M(|u|^2) = 0\},$$

M being the unique invariant mean on $B(G)$, is a closed ideal in $B(G)$. Moreover, we have the direct decomposition

$$B(G) = B_c(G) \oplus \big(\mathrm{AP}(G) \cap B(G)\big),$$

where $\mathrm{AP}(G)$ is a space of almost periodic functions on G.

2. Structure of the $C_\Delta^*(F)$. We start with the following

LEMMA 1. *Let $\pi_1, \pi_2 \in \Sigma(G)$, $\pi_1 \prec^{\mathrm{w}} \pi_2$ and suppose that $C_{\pi_1}^*(G)$ is a simple algebra. Then π_1 is weakly equivalent to π_2.*

Proof. Since $\pi_1 \prec^{\mathrm{w}} \pi_2$, $N'_{\pi_2} \subset N'_{\pi_1}$ and the diagram

$$\begin{tikzcd} C_{\pi_1}^*(G) \arrow[dr, dashed, "\beta"'] & C^*(G) \arrow[l] \arrow[d] \\ & C_{\pi_2}^*(G) \end{tikzcd}$$

is commutative, where β is such that $a_{\pi_2} = \beta \circ a_{\pi_1}$. A simple verification shows that β is well defined and that it is a $*$-epimorphism.

But since $C_{\pi_1}^*(G)$ is simple and $\ker\beta$ is trivial, β is a $*$-isomorphism. Consequently, $C_{\pi_1}^*(G)$ is $*$-isomorphic to $C_{\pi_2}^*(G)$, so $B_{\pi_1}(G) = B_{\pi_2}(G)$ and π_1 is weakly equivalent to π_2.

COROLLARY 1. *Let F be a non-commutative free group. Suppose that $\pi \in \Sigma$ is weakly contained in the left regular representation ϱ. Then π is weakly equivalent to ϱ.*

Proof. It follows from Lemma 1 and a theorem of Powers [15] (cf. [2]) that $C_\varrho^*(F)$ is a simple algebra.

Now, let H be a subgroup of F and let $U^H = \mathrm{ind}_H^G 1$ be the induced representation of F from the trivial representation of the subgroup H. We call it a *quasi-regular representation*.

Let $x \in F$ and let $C(x)$ denote the centralizer of x. We know that $C(x)$ is a cyclic subgroup of the free group F (cf., e.g., [13], p. 42).

We write $U^{C(x)} = U^x$ and we note that U^e is a trivial representation of F. The following is an extension of a theorem of Yoshizawa [16].

PROPOSITION 1. *Let F be a non-commutative free group. Then the representation U^x is irreducible and, for $x \neq e$, U^x is weakly equivalent to the left regular representation λ.*

Proof. The quasi-regular representation U^x acts in the Hilbert space $l^2(F/C(x))$ by the formula

$$(U_g^x f)(\bar{z}) = f(\overline{g^{-1}z}),$$

where $f \in l^2(F/C(x))$ and $\bar{z} = zC(x)$. To simplify the notation, we denote the function $\delta_{\bar{z}}$ by $\bar{z}$.

First we prove

$$(1) \qquad 0 \neq f \in l^2(F/C(x)) \quad \text{and} \quad (U_x^x f = f) \Rightarrow (f = f(\bar{e})\bar{e}).$$

In fact, $U_x^x f = f$ implies $U_{x^k}^x f = f$ for every integer k, and hence

$$(2) \qquad f(\overline{x^k y}) = f(\bar{y}) \quad \text{for every } \bar{y} \in F,\, k \in Z.$$

Since the centralizer of x is an infinite cyclic subgroup, we see that the set of the cosets $\{\overline{x^k y}\colon k \in Z\}$ is infinite provided $y \in C(x)$. Thus, by (2), $f = 0$ for $\bar{y} \neq C(x)$, whence $f = f(\bar{e})\bar{e}$.

What follows now is an adaptation of the proof due to Yoshizawa [16].

Let T be an intertwining operator of the representation U^x, i.e.

$$TU_g^x = U_g^x T \quad \text{for every } g \in F.$$

Hence, in particular, we have

$$U_x^x T(\bar{e}) = TU_x^x(\bar{e}) = T(\bar{e}).$$

So, by (1), we obtain $T(\bar{e}) = a\bar{e}$. But

$$T\bar{x} = TU_x^x(\bar{e}) = U_x^x T(\bar{e}) = a\bar{x},$$

whence $T = aI$ and, consequently, U^x is an irreducible representation.

By Lemma 1, to show that U^x is weakly equivalent to ϱ it suffices to prove that U^x is weakly contained in ϱ. This, however, results from the following well-known theorems:

THEOREM B (Fell). *If H is a closed subgroup of G and $\pi_1 \prec^{w} \pi_2$, then*

$$\mathrm{ind}_H^G(\pi_1) \prec^{w} \mathrm{ind}_H^G(\pi_2).$$

THEOREM C (Mackey). *Let $H_1 \subset H_2 \subset G$ and $\pi \in \Sigma(H_1)$. Then*

$$\mathrm{ind}_{H_1}^G(\pi_1) \cong \mathrm{ind}_{H_2}^G(\mathrm{ind}_{H_1}^{H_2}(\pi_1)).$$

THEOREM D (Hulanicki). *A locally compact group G is amenable if and only if the trivial representation of G is weakly contained in ϱ.*

Since the centralizer $C(x)$ of x is a cyclic group, so amenable, we have

$$i_{C(x)} \prec^{w} \varrho_{C(x)} = \mathrm{ind}_{\{e\}}^{C(x)}(i_{\{e\}}).$$

But $U^x = \mathrm{ind}_{C(x)}^{F}(i_{C(x)})$, whence, by Theorems B and C, Proposition 1 follows.

We are now in a position to consider the conjugate representation Δ of the free group F and to prove a theorem describing the structure of $C^*_\Delta(F)$. The representation Δ acts in $l^2(F)$ by the following formula:

$$(\Delta_g f)(x) = f(g^{-1}xg) \quad \text{for } f \in l^2(F).$$

Let $O(x) = \{g^{-1}xg\colon\, g \in F\}$. Then

$$F = \bigcup_{x \in R} O(x),$$

where x runs over a set R such that $x, y \in R$ implies $O(x) \cap O(y) = \emptyset$.

THEOREM 1. *Let F be a non-commutative free group and let Δ be the conjugate representation of F. Then*

(a) *Δ is unitarily equivalent to the representation $\bigoplus_{x \in R} U^x$.*

(b) *Δ is a faithful *-representation of $l^1(F)$ (i.e., if $\Delta(f) = 0$ for an $f \in l^1(F)$, then $f = 0$).*

(c) *$C^*_\Delta(F)$ contains the central projection P on the one-dimensional subspace $\mathscr{H}_1 = \{\beta\delta_e\colon\, \beta \in C\}$.*

(d) *$C^*_\Delta(F) = I_1 \oplus I_2$, where I_1 and I_2 are unique closed ideals in $C^*_\Delta(F)$ such that $I_1 \cap I_2 = \{0\}$, I_1 is isomorphic to the complex numbers C, and I_2 is isometrically isomorphic to $C^*_\varrho(F)$.*

Proof. (a) Since $F = \bigcup_{x \in R} O(x)$, we have

$$l^2(F) = \bigoplus_{x \in R} l^2\big(O(x)\big).$$

To prove (a) it suffices to note that the representation U^x is unitarily equivalent to $\Delta|_{l^2(O(x))}$, and this is established by the map

$$\gamma_x\colon\, O(x) \to F/C(x)$$

defined by $\gamma_x(yxy^{-1}) = yC(x)$.

(b) By Proposition 1 and Theorem 1 (a), the representation Δ is weakly equivalent to $i_G \oplus \varrho$. Hence

$$\varrho \prec^{w} \Delta$$

and this is equivalent to

$$\|\varrho(f)\| \leqslant \|\Delta(f)\| \quad \text{for every } f \in l^1(F).$$

The left regular representation ϱ is faithful on $l^1(F)$ and so is Δ.

By (b), the C^*-algebra $C^*_\Delta(F)$ can be regarded as the closure of the algebra of operators $\{\Delta(f): f \in l^1(F)\}$ in the space of all bounded operators on the Hilbert space $l^2(F)$.

(c) In fact, (c) was first noticed by Akemann and Ostrand [1] in a similar context.

For the sake of completeness we include our proof of (c).

It depends essentially on the existence of an infinite subset E in a non-commutative free group which has the following property:

$$\|\varrho(f)\| \leqslant C\|f\|_2 \quad \text{for } f \in l^2(F) \text{ and } \operatorname{supp} f \subset E.$$

The existence of such an infinite set in a free group has been discovered by Leinert [11] and, therefore, it is called *Leinert set* (see [12] and [3]).

If $E = \{a^n b^n: n = 1, 2, \ldots\}$, where a and b are free elements in a free group F, then E is a Leinert set.

Let P denote the projection in $l^2(F)$ onto the one-dimensional subspace $\mathscr{H}_e = \{\beta\delta_e: \beta \in C\}$:

$$Pf = f(e)\,\delta_e \quad \text{for every } f \in l^2(F).$$

Let $E = \{x_1, x_2, \ldots\}$ be an infinite Leinert set. We put

$$f_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{x_k}.$$

We evaluate the operator norm of the difference $\Delta(f_n) - P$.

Using Proposition 1, statement (a), and the fact that E is a Leinert set we obtain

$$\begin{aligned}\|\Delta(f_n) - P\| &= \big\|\Delta(f_n)|_{x\neq e} \oplus l^2\big(O(x)\big)\big\| = \sup_{x\neq e}\|U^x(f_n)\| \\ &\leqslant \|\varrho(f_n)\| \leqslant C\|f_n\|_2 = Cn^{-1/2}.\end{aligned}$$

It is also easy to see that

$$P\Delta(f) = \Delta(f)P = P(f) \quad \text{for every } f \in l^1(F)$$

and that P belongs to the centrum of the algebra $C^*_\Delta(F)$.

(d) Making use of the formula

$$\|(I-P)\Delta(f)\| = \|\varrho(f)\|$$

which is a consequence of Proposition 1 and Theorem 1 (a), we note that the mapping

$$\Delta(f) \to \big(P(f),\ (I-P)\Delta(f)\big)$$

establishes a *-isomorphism between the algebras $C^*_\Delta(F)$ and $C\cdot 1 \oplus C^*_\varrho(F)$.

The simplicity of $C^*_\varrho(F)$ implies that $C^*_\Delta(F)$ has only two ideals I_1 and I_2. Since the centrum of the C^*-algebra $C^*_\varrho(F)$ is trivial, we see that I_1 is the centrum of $C^*_\Delta(F)$.

COROLLARY 2. $B_\Delta(F) = B_\varrho(F) \oplus C\cdot 1$.

3. Special subalgebra of the Fourier-Stieltjes algebra $B(F)$. Now let G be an arbitrary locally compact group. As has been shown by Godement [9], there exists a unique invariant mean M on $B(G)$.

In particular, if $u \in P(G)$, then

$$M(u) = \inf\{\langle u, f * f^*\rangle : f \geqslant 0, \|f\|_1 = 1\}.$$

Let

$$B_c(G) = \{u \in B(G) : M(|u|^2) = 0\}.$$

We have a direct decomposition

$$B(G) = B_c(G) \oplus B_d(G).$$

PROPOSITION E (Derighetti). *If G is non-amenable, then $B_\varrho(G) \subset B_c(G)$.*

For the sake of completeness we include a modified proof of Derighetti [5]:

Proof. Let $u \in P_\varrho(G)$. Then $u = g * g^*$ for some $g \in L^2(G)$.

Since G is not amenable, there exists an $f \in L^1(G)$ such that $f \geqslant 0$, $\|f\|_1 = 1$ and $\|\varrho(f)\| = k < 1$. We put $h_n = f * f * \ldots * f$ (n times). Of course, $\|h_n\|_1 = 1$, $\|\varrho(h_n)\| < k^n$ and, consequently,

$$|M(u)| \leqslant \langle u, h_n * h_n^*\rangle = \langle g * g^*, h_n * h_n^*\rangle \leqslant \|\varrho(h_n)\|^2 \|g\|_2^2.$$

Since $\|\varrho(h_n)\| \to 0$, we obtain $M(u) = 0$. Therefore, if $u \in P_\varrho(G)$, then $u\bar{u} = |u|^2 \in P_\varrho(G)$, whence $M(|u|^2) = 0$ and, consequently,

$$M(|u|^2) = 0 \quad \text{for every } u \in B_\varrho(G).$$

PROBLEM (**P 1074**). Does $B_\varrho(G) = B_c(G)$ hold for any locally compact group G?

We are going to show that this does not hold for non-commutative free groups.

PROPOSITION 2. *If F is the free group freely generated by two elements a and b, then $B_\varrho(F) \subsetneqq B_c(F)$.*

First proof (Hulanicki). Let N be the smallest normal subgroup of the free group F containing the free generator a. Denote by χ the characteristic function of N. Since the index of N in F is infinite ($F/N \cong Z$), we have $\chi \in B_c(F)$. If $\chi \in B_\varrho(F)$, then $\chi|_N \in B_\varrho(N)$. But $\chi|_N \equiv 1$, and N is also a free group. Thus N is non-amenable and, by Theorem D, $1 \notin B_\varrho(N)$, so also $\chi \notin B_\varrho(F)$.

Second proof. Let χ be as in the first proof. We show that $\chi \notin B_\varrho(F)$ using the following theorem about Leinert sets (see [14]):

If E is a Leinert set, then

$$\chi_E B_\varrho(F) = l^2(E). \tag{3}$$

One can verify that $E = \{b^k a b^{-k}: k = 1, 2, \ldots\}$ is a Leinert set and $E \subset N$. Thus, by (3), we obtain $\chi \notin B_\varrho(F)$.

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