COLLOQUIUM MATHEMATICUM

VOL. LVII 1989 FASC. 2

ON RIGID SUBSETS OF SOME MANIFOLDS

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Borsuk was concerned in [2] and [3] with some problems in intrinsic geometry of Euclidean spaces. In particular, his Theorem (3.1) in [3] claims that if X is an open connected subset of the Euclidean n-space E^n, then every intrinsic isometry

$$f\colon X \to f(X) \subset E^n$$

is an isometry.

A special case of this result (for $X = E^n$) was generalized to a certain class of n-manifolds by Herburt and the author in [7].

The purpose of the present paper is to generalize Theorem (3.1) of [3] to a possibly large class of n-manifolds (cf. 2.1).

0. Preliminaries. We follow the terminology and notation of [7]. In particular, we refer to a metric space $M = (M, \varrho)$ as *strongly arcwise connected* whenever every two distinct points of M can be joined in M by an arc of finite length; for a strongly arcwise connected space (M, ϱ) the *intrinsic metric* ϱ^* *induced by* ϱ is defined by the formula

$$\varrho^*(x, y) := \inf\{|L|;\ L \text{ is an arc in } M \text{ and } x, y \in L\}.$$

The class GA of *geometrically acceptable* metric spaces introduced by Borsuk in [2] can be equivalently defined as follows:

$$\mathrm{GA} := \{M = (M, \varrho);\ M \text{ is strongly arcwise connected and } \varrho^* \text{ is topologically equivalent to } \varrho\}$$

(cf. [2], [3], [7]).

Let us notice that

0.1. *If $f\colon X \to Y$ is a homeomorphism preserving the lengths of arcs, then*

$$X \in \mathrm{GA} \Rightarrow Y \in \mathrm{GA}.$$

Proof. Clearly, if X is strongly arcwise connected, then so is Y. Let, moreover, ϱ_X and $(\varrho_X)^*$ be topologically equivalent:

$$\varrho_X \equiv (\varrho_X)^*.$$

Then for every sequence $(x_n)_{n\in N}$ in X and $x_0 \in X$ the following two conditions are equivalent:

(1) $$\lim_n \varrho_X(x_n, x_0) = 0,$$

(2) $$\lim_n \inf\{|L_n|;\ x_n, x_0 \in L_n\} = 0.$$

Since f is a homeomorphism, (1) is equivalent to

(1′) $$\lim_n \varrho_Y(f(x_n), f(x_0)) = 0.$$

Since f preserves the lengths of arcs, (2) is equivalent to

(2′) $$\lim_n \inf\{|f(L_n)|;\ f(x_n), f(x_0) \in f(L_n)\} = 0.$$

Thus (1′) is equivalent to (2′), whence $\varrho_Y \equiv (\varrho_Y)^*$. This completes the proof.

Let $(X, \varrho_X) \in \mathrm{GA}$ and $(Y, \varrho_Y) \in \mathrm{GA}$. A surjective map $f\colon X \to Y$ is an *intrinsic isometry* of (X, ϱ_X) onto (Y, ϱ_Y) whenever f is an isometry of $(X, (\varrho_X)^*)$ onto $(Y, (\varrho_Y)^*)$. Since, by Theorem (2.1) of [3], a map f is an intrinsic isometry if and only if f is a homeomorphism preserving the lengths of arcs, Proposition 0.1 justifies the following

0.2. Definition. Let $\boldsymbol{M} = (M, \varrho)$ be a metric space and X its geometrically acceptable subspace(¹). The set X is *rigid in* $\boldsymbol{M}$ if and only if every intrinsic isometry $f\colon X \to f(X) \subset M$ is an isometry(²).

Using this terminology, we can reformulate (3.1) of [3] as follows:

0.3. *Every open connected subset of the Euclidean n-space* E^n *is rigid in* E^n.

In the sequel we shall need the following well-known notions: A metric space $\boldsymbol{M} = (M, \varrho)$ is (*strongly*) *convex* whenever every pair of distinct points $x, y \in M$ can be joined by a (unique) segment L in M, i.e., there is a (unique) arc $L \subset M$ with end-points x, y isometric to the interval $[0, \varrho(x, y)] \subset R$(³). The space $\boldsymbol{M}$ is *locally convex* if and only if for every $x \in M$ and every neighbourhood U_x of x in $\boldsymbol{M}$ there is a convex neighbourhood $V_x \subset U_x$. Further, $\boldsymbol{M}$ is (*metrically*) *homogeneous with respect to a family* $\mathscr{X} \subset 2^M$ whenever for every $A, B \in \mathscr{X}$ every isometry f_0 of $(A, \varrho|A^2)$ onto $(B, \varrho|B^2)$ can be extended to an isometry f of the whole space $\boldsymbol{M}$. The space $\boldsymbol{M}$ is (*metrically*) *perfectly homogeneous* whenever it is homogeneous with respect to 2^M. Finally, let B_ϱ be the *metric betweenness relation* (⁴), i.e.,

$$B_\varrho(a, b, c) \Leftrightarrow \varrho(a, b) + \varrho(b, c) = \varrho(a, c).$$

(¹) I. e., more precisely, $(X, \varrho|X^2) \in \mathrm{GA}$.

(²) For other notions of rigidity see [5].

(³) For complete spaces this notion of convexity coincides with that in the sense of Menger (cf. [1], p. 41). For G-spaces in the sense of Busemann a different notion of convexity was introduced in [4].

(⁴) It was studied in [8].

The space M *has no ramifications* whenever B_ϱ satisfies the following condition: for every $p, x, a, b \in M$

$$p \neq x \wedge B_\varrho(p, x, a) \wedge B_\varrho(p, x, b) \Rightarrow B_\varrho(x, a, b) \vee B_\varrho(x, b, a) \ (^5).$$

The space M has *uniquely prolongable segments* whenever for every $p, x, a, b \in M$

$$p \neq x \wedge B_\varrho(p, x, a) \wedge B_\varrho(p, x, b) \wedge \varrho(x, a) = \varrho(x, b) \Rightarrow a = b \ (^6)$$

(cf. [4], p. 36).

Clearly:

0.4. *If M has no ramifications, then it has uniquely prolongable segments.*

1. Geometric subspaces. Given a metric space M, we shall define an operation

$$\mathrm{Dist}_M\colon 2^M \to 2^M$$

assigning to a subset A of M the set $\mathrm{Dist}_M(A)$ of points which are uniquely determined by their distances from all the points of A (Definition 1.3). Let us first generalize the notion of bisectrix.

1.1. DEFINITION. For any $a, b \in M$, let

$$L_M\binom{a}{b} = \{x \in M;\ \varrho(x, a) = \varrho(x, b)\}.$$

We refer to $L_M\binom{a}{b}$ as the *bisectrix of the pair* (a, b) *in* M.

Evidently:

1.2. *For every* $a, b \in M$

(i) $L_M\binom{a}{b} = L_M\binom{b}{a}$;

(ii) $L_M\binom{a}{b} = M$ *if and only if* $a = b$.

1.3. DEFINITION. For any $A \subset M$, let

$$\mathrm{Dist}_M(A) = \left\{x \in M;\ \forall y \in M \left(A \subset L_M\binom{x}{y} \Rightarrow x = y\right)\right\}.$$

We refer to $\mathrm{Dist}_M(A)$ as the *geometric subspace of* M *generated by* A.

Clearly:

1.4. (i) $A \subset \mathrm{Dist}_M(A)$ *for every* $A \subset M$;

(ii) $A \subset B \Rightarrow \mathrm{Dist}_M(A) \subset \mathrm{Dist}_M(B)$ *for every* $A, B \subset M$.

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(5) In [9], p. 111, this property is referred to as *outer smoothness* II.

(6) Notice that the "segment" (p, x) does not have to be prolongable, but if it is, then is prolongable uniquely.

1.5. EXAMPLES. (a) Let $M = E^n$. Then $\mathrm{Dist}_M = Af$, i.e., for every $A \subset E^n$ the set $\mathrm{Dist}_M(A)$ is the affine subspace spanned by A.

(b) Let $S^n = \{x \in R^{n+1};\ \|x\| = 1\}$ and let ϱ be the Euclidean metric restricted to S^n. If $M = (S^n, \varrho)$ or $M = (S^n, \varrho^*)$, then for every $A \subset S^n$ the set $\mathrm{Dist}_M(A)$ is the smallest unit subsphere of S^n containing A.

By 1.5, if M is the n-sphere or the Euclidean n-space, then every ball B in M generates the whole space M, i.e., $\mathrm{Dist}_M(B) = M$. Let us prove

1.6. PROPOSITION. *If M is convex and has uniquely prolongable segments, then*

$$\mathrm{Dist}_M(B) = M$$

for every ball B in M.

Proof. We shall prove

(1)
$$\forall a, b \in M \left(a \neq b \Rightarrow \mathrm{Int}\, L_M\binom{a}{b} = \emptyset \right).$$

Let $a \neq b$. Since M is convex, $L_M\binom{a}{b} \neq \emptyset$; let

$$p \in L_M\binom{a}{b}.$$

There exist metric segments L' and L'' joining a with p and b with p, respectively. Let us show that

(2)
$$L' \cap L_M\binom{a}{b} = \{p\} = L'' \cap L_M\binom{a}{b}.$$

Indeed, let $x \in L' \cap L_M\binom{a}{b}$; then

(3)
$$B_\varrho(p, x, a) \quad \text{and} \quad \varrho(a, x) = \varrho(x, b),$$

which, together with $\varrho(a, p) = \varrho(p, b)$, implies

(4)
$$B_\varrho(p, x, b).$$

Since M has uniquely prolongable segments and $a \neq b$, (3) and (4) imply $p = x$. This proves (2). By (2), there is a sequence $(x_n)_{n \in N}$ such that

$$x_n \in L' - L_M\binom{a}{b} \quad \text{and} \quad \lim_n x_n = p.$$

This proves (1).

Suppose now there is a ball B_0 with

$$\mathrm{Dist}_M(B_0) \neq M.$$

Then

$$\exists a, b \in M \left(a \neq b \wedge B_0 \subset L_M\binom{a}{b} \right),$$

contrary to (1).

1.7. COROLLARY. *If $M = (M, \varrho)$ is a complete 3-smooth n-dimensional manifold with $\varrho = \varrho^*$ (or, more generally, a G-space in the sense of Busemann* [4], *then $\mathrm{Dist}_M(B) = M$ for every ball B in M.*

2. Manifolds with open connected subsets being rigid. Let us establish the main result. Its proof is a generalization of the proof of 3.1 in [3].

2.1. THEOREM. *Let M be a finite dimensional manifold. If*

(i) *M is strongly arcwise connected,*

(ii) *M is locally convex,*

(iii) *M is homogeneous with respect to a family of sufficiently small balls,*

(iv) *for every ball B in M*

$$\mathrm{Dist}_M(B) = M,$$

then every open connected subset of M is rigid in M.

Proof. By (i) and (ii), $M \in \mathrm{GA}$. Take an open connected set $X \subset M$. If $X = \emptyset$, then X is rigid. Let $X \neq \emptyset$ and let $h\colon X \to Y \subset M$ be an intrinsic isometry. Notice that for every $A, B \subset M$

(1) *if $A \subset X$, $B \subset Y$, $h(A) \subset B$, and A, B are convex, then $h|A$ is an isometric embedding of $(A, \varrho|A^2)$ into $(B, \varrho|B^2)$.*

Indeed, let $A \subset X$, $B \subset Y$, $h(A) \subset B$, and let A, B be convex. Then

$$\forall x_1, x_2 \in A \ \varrho(x_1, x_2) = (\varrho|X^2)^*(x_1, x_2) = (\varrho|Y^2)^*(h(x_1), h(x_2)) = \varrho(h(x_1), h(x_2)),$$

which proves (1).

Let us prove

(2) *for every $x_0 \in X$ there exist convex neighbourhoods U_{x_0} and $V_{h(x_0)}$ such that $h(U_{x_0}) \subset V_{h(x_0)}$.*

Indeed, h is a homeomorphism, so Y is open in M. Let $x_0 \in X$ and $y_0 = h(x_0)$; there is a $\delta > 0$ with open ball $B_M(y_0, \delta)$ contained in Y. By (ii), there is a convex neighbourhood V_{y_0} contained in the ball $B_M(y_0, \delta)$. Since h is continuous, by (ii) there is a convex U_{x_0} such that $h(U_{x_0}) \subset V_{y_0}$, which proves (2).

By (1) and (2), there is an open subset U of X satisfying the condition

(3) $h|U$ is an isometry.

By (iv) and 1.4 (ii), $\mathrm{Dist}_M(U) = M$, whence, by (iii) and (3), there is a $g \in \mathrm{Aut}\, M$ such that

(4) $$g|U = h|U.$$

Let

$$X_0 = \{x \in X;\ g(x) = h(x)\}.$$

Evidently, $U \subset X_0 \subset X$, whence

(5) $$\operatorname{Int} X_0 \neq \emptyset.$$

It remains to prove that $X_0 = X$. Suppose to the contrary that $X_0 \neq X$. Since X is connected, it follows by (5) that

$$\operatorname{Fr}_X \operatorname{Int} X_0 \neq \emptyset.$$

Let $b \in \operatorname{Fr}_X \operatorname{Int} X_0$. Then $b \in \operatorname{Fr}_X X_0$. By (2), there exist convex neighbourhoods U_b and $V_{h(b)}$ such that $h(U_b) \subset V_{h(b)}$. Clearly,

(6) $$U_b \cap \operatorname{Int} X_0 \neq \emptyset\cdot$$

and

(7) $$U_b \cap (X - X_0) \neq \emptyset.$$

By (iv) and (6), there is $C \subset U_b \cap X_0$ such that

(8) $$\operatorname{Dist}_M(C) = M.$$

By (1) it follows that $h|U_b$ is an isometry. Since $C \subset X_0$, we have $h|C = g|C$. Therefore

(9) $$\forall x \in U_b\ \forall a \in C\ \ \varrho(x, a) = \varrho(h(x), g(a)).$$

On the other hand,

$$\forall x \in U_b\ \forall a \in C\ \ \varrho(x, a) = \varrho(g(x), g(a)),$$

whence, by (9), we obtain

(10) $$\forall x \in U_b\ \ g(C) \subset L_M\begin{pmatrix} h(x) \\ g(x) \end{pmatrix}.$$

Evidently, (8) implies

$$\operatorname{Dist}_M(g(C)) = M,$$

whence, by (10), $h|U_b = g|U_b$, i.e., $U_b \subset X_0$, contrary to (7).

By 2.1 and 1.7 we obtain the following two analogues of the Borsuk theorem (cf. 0.3):

2.2. COROLLARY. *Every open connected subset of S^n is rigid in (S^n, ϱ^*) as well as in (S^n, ϱ)* (cf. 1.5(b)).

P r o o f. For (S^n, ϱ^*) this is a consequence of 2.1 and 1.7. Since neither the class of intrinsic isometries nor the class of isometries differs for (S^n, ϱ^*) and (S^n, ϱ), the second part of the statement follows from the first one.

2.3. COROLLARY. *Let T^2 be a 2-dimensional torus of revolution in R^3 and let*

ϱ^* be induced by the Euclidean metric ϱ. Then every open and connected subset of T^2 is rigid in (T^2, ϱ^*).

Proof. It suffices to check conditions (i)–(iv) of 2.1. Evidently, (T^2, ϱ^*) satisfies (i) and (ii); by 1.7 it satisfies (iv). To show that it satisfies (iii), let us consider two isometric balls $B_i = B_{T^2}(a_i, \varepsilon)$, $i = 1, 2$, with arbitrary centres a_1, a_2 and sufficiently small radius $\varepsilon > 0$ (smaller than one half of the radius of meridians). Since ϱ^* is the intrinsic metric, there is an intrinsic isometry f: $B_1 \to B_2$. The map f preserves Gauss curvature K. Since K is constant along each latitude and since two latitudes A_1, A_2 have the same K if and only if either $A_1 = A_2$ or A_1 and A_2 are symmetric with respect to the equatorial plane P, for each latitude A either $f(B_1 \cap A) = B_2 \cap A$ or $f(B_1 \cap A) = B_2 \cap A'$, where A' is symmetric to A. Moreover, f preserves the angles between arcs on T^2 and preserves the family of geodesics. Thus, it preserves the family of meridians, so for each meridian C_1 there is a meridian C_2 such that

$$f(B_1 \cap C_1) = B_2 \cap C_2.$$

Consequently, f can be extended to either a rotation of R^3 or the composition of a rotation and the reflection at P. Thus (iii) is satisfied.

The following example (due to the referee of the first version of this paper) shows that Theorem 2.1 cannot be generalized to infinite dimensional manifolds.

2.4. EXAMPLE. Let M be the Hilbert space, $M = R^\omega$ with the standard metric ϱ: if

$$x = (x_1, x_2, \ldots), \qquad y = (y_1, y_2, \ldots),$$

then

$$\varrho(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}.$$

The map f: $M \to f(M) \subset M$ defined by the formula

$$f(x_1, x_2, x_3, \ldots) = \begin{cases} (x_1, 0, x_2, x_3, \ldots) & \text{if } x_1 \leqslant 0, \\ (0, x_1, x_2, x_3, \ldots) & \text{if } x_1 \geqslant 0 \end{cases}$$

is an intrinsic isometry which is not an isometry.

It is worthy of notice that Theorem 2.1 remains valid for every metric space M which satisfies (i)–(iv) and has the domain invariance property.

3. **Final remarks.** Various kinds of rigidity can be found in the literature (for references see, e.g., [5]). Most of them are particular cases of one of the following two notions.

3.1. DEFINITION. Let $\mathscr{F}$ be a family of intrinsic isometries of subsets of a metric space $M = (M, \varrho)$. A set $X \subset M$ is *rigid* (*weakly rigid*) *in M with respect to $\mathscr{F}$* if and only if, for every f: $X \to Y \subset M$,

$$f \in \mathscr{F} \Rightarrow f \text{ is an isometry } (X \text{ is isometric to } Y).$$

Clearly, if $\mathscr{F}$ is the family of all intrinsic isometries, then the rigidity with respect to $\mathscr{F}$ coincides with the rigidity in the sense of Definition 0.2; similarly, the weak rigidity with respect to $\mathscr{F}$ is then referred to as *weak rigidity*.

Both the rigidity and weak rigidity, especially for subsets of Euclidean spaces, have been studied by Herburt in her doctoral dissertation [6].

The following statement answers the question under what assumption on a metric space (X, ϱ) the weak rigidity of X in arbitrary overspace M is equivalent to rigidity of X in M.

3.2. PROPOSITION. *Let $\mathscr{F}$ be a family of intrinsic isometries, which is closed under composition and contains all isometries. Then for every metric space (X, ϱ) the following two conditions are equivalent*:

(α) *X is rigid in itself with respect to $\mathscr{F}$*;

(β) *for every overspace M of X, if X is weakly rigid in M with respect to $\mathscr{F}$, then X is rigid in M with respect to $\mathscr{F}$.*

Proof. $(\alpha)\Rightarrow(\beta)$. Let X be weakly rigid in M with respect to $\mathscr{F}$ and let $g\colon X \to g(X) \subset M$ be an intrinsic isometry, $g \in \mathscr{F}$. Then there exists an isometry $h\colon g(X) \to X$. Let $f = h \circ g$. Clearly, f is an intrinsic isometry of X onto itself and $f \in \mathscr{F}$. Thus, by (α), the map f is an isometry, and therefore g is an isometry.

$(\beta)\Rightarrow(\alpha)$. Take $M = X$. Evidently, X is weakly rigid in itself; in particular, it is weakly rigid in itself with respect to $\mathscr{F}$, whence, by (β), X is rigid in itself with respect to $\mathscr{F}$.

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Reçu par la Rédaction le 23.3.1987;
en version modifiée le 4.11.1987