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DIAGONAL-LIKE EMBEDDINGS

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Let $\mathfrak{A}$ be an axiomatic class of relational systems. A projective subclass of $\mathfrak{A}$ is, roughly speaking, the domain of an axiomatic correspondence between the models $\boldsymbol{A} \in \mathfrak{A}$ and $\boldsymbol{B} \in \mathfrak{B}$, where $\mathfrak{B}$ is an arbitrary axiomatic class. More precisely, let L_ϱ and L_σ be first-order languages for $\mathfrak{A}$ and $\mathfrak{B}$, respectively. The indices ϱ and σ denote the similarity types of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. We may assume that L_ϱ and L_σ have only the logical symbols in common. We extend the union $L_\varrho \cup L_\sigma$ by two one place relational symbols R_A and R_B to $L_{[\varrho,\sigma]}$. In $L_{[\varrho,\sigma]}$ there exists a set $K_{\mathfrak{A},\mathfrak{B}}$ such that

$$\boldsymbol{C} \in \mathrm{Md}(K_{\mathfrak{A},\mathfrak{B}}) \Longleftrightarrow \boldsymbol{C} = \boldsymbol{A} \dot{\cup} \boldsymbol{B}, \quad \text{where } \boldsymbol{A} \in \mathfrak{A}, \boldsymbol{B} \in \mathfrak{B}.$$

The $\mathfrak{A}$-components of certain pseudo-axiomatic subclasses of $\mathfrak{L} = \mathrm{Md}(K_{\mathfrak{A},\mathfrak{B}})$ are called *projective subclasses* of $\mathfrak{A}$. Extend $L_{[\varrho,\sigma]}$ by a set Θ of relational symbols T to L_τ and call a set K of sentences in L_τ *admissible* if

1. $K \vdash K_{\mathfrak{A},\mathfrak{B}}$,
2. $K \vdash \bigwedge_{x_1} \dots \bigwedge_{x_n} T(x_1 \dots x_n) \to R_{C_1}(x_1) \wedge \dots \wedge R_{C_n}(x_n)$ for $T \in \Theta$ and for certain $C_i \in \{A, B\}$ depending on T.

An admissible set K defines an axiomatic relation, or correspondence, Γ between $\mathfrak{A}$ and $\mathfrak{B}$ such that $\boldsymbol{A}\Gamma\boldsymbol{B}$ holds if and only if under a suitable interpretation of the relations $T \in \Theta$ on $\boldsymbol{A} \dot{\cup} \boldsymbol{B}$ one gets a model $(\boldsymbol{A} \dot{\cup} \boldsymbol{B})^\wedge$ of K. A projective subclass $\mathscr{P}$ of $\mathfrak{A}$ is defined by

$$\boldsymbol{A} \in \mathscr{P} \text{ iff } (\boldsymbol{A} \dot{\cup} \boldsymbol{B})^\wedge \models K \text{ for some } \boldsymbol{B} \in \mathfrak{B}.$$

Projective classes were introduced by Mal'cev [6]; his definition is even somewhat more general; they are closed under ultraproducts and ultralimits (cf. Armbrust and Kaiser [2] and [3]), but in general not closed under elementary extensions. As a very simple example, we consider the class of all discretely ordered sets with first element but without last element. The order relation $<$, the first element 0, and a binary relation S, indicating the successor of an element, belong to the type of $\mathfrak{A}$. Let $\mathfrak{B}$ be the axiomatic class of all elementary extensions of the ordered semigroup $N^+ = (N, +, 0, <, S)$. $\mathfrak{B}$ is a class of models of the additive number theory. As the axiomatic relation Γ between $\mathfrak{A}$ and $\mathfrak{B}$ we take: A admits a semigroup operation and the extended structure is a model $A^\wedge \in \mathfrak{B}$. This projective subclass $\mathscr{P}$ of $\mathfrak{A}$ contains the natural numbers N, but not the ordered sum $N_1 = N \dot{\cup} Z$ $(N < Z)$.

The only possible embedding $\varphi: N \to N_1$ is elementary (cf. Robinson [7], p. 98), but $N_1 \notin \mathscr{P}$. This example shows us that the connection between N and N_1 is weaker than the connection between N and any of its ultrapowers or ultralimits.

Let A and A^* be similar structures, $A, A^* \in \mathfrak{A}_\varrho$, where $\mathfrak{A}_\varrho$ is the similarity class for the type ϱ. We say that A^* is *elementarily lifted over* A, $A < A^*$, if for any projective class $\mathscr{P}$ one has $A \in \mathscr{P} \to A^* \in \mathscr{P}$.

We have $A < A^I/\mathfrak{u}$ and $A < A_\lambda$ for any ultrapower $A^I/\mathfrak{u}$ and ultralimit A_λ, respectively, over A. If A^* is elementarily lifted over A, then A^* is an elementary extension of A. Let $\mathfrak{B}$ be the class of all elementary extensions of A and let the axiomatic relation Γ be given by the isomorphism relation with respect to ϱ. We remark that to the type of $\mathfrak{B}$ the elements of A are added as constants to have $\mathfrak{B}$ as an axiomatic class.

The aim of this paper is to show that $A < A^*$ not only implies the existence of an elementary map $\varphi: A \to A^*$, but one of an extremely special kind. Let A and A_1 be similar structures taken from an axiomatic class $\mathfrak{A}$. We call an elementary map $\delta: A \to A_1$ *diagonal-like* if for any axiomatic relation Γ between $\mathfrak{A}$ and an axiomatic class $\mathfrak{B}$ one has for any realization of K,

$$(A \dot{\cup} B)^\wedge \models K,$$

a model $B_1 \in \mathfrak{B}$, and an interpretation of the relations $T \in \Theta$ such that

$$(A_1 \dot{\cup} B_1)^\wedge \models K$$

and, moreover, δ can be lifted to an elementary embedding

$$\delta^\wedge : (A \dot{\cup} B)^\wedge \to (A_1 \dot{\cup} B_1)^\wedge .$$

For example, the diagonals $\Delta: A \to A^I/\mathfrak{u}$ and $\Delta: A \to A_\lambda$ are diagonal-like. To illustrate how a diagonal-like map works let $\mathfrak{A} = \mathfrak{A}_\varrho$ and let $\mathfrak{B}$ be an axiomatic class of extended type σ. Let Γ stand for: A is a ϱ-reduct of an $A^\wedge$ in $\mathfrak{B}$. K expresses the fact that there is a ϱ-isomorphism $\varphi: A \to A^\wedge$. If $\delta: A \to A_1$ is diagonal-like, we have a ϱ-isomorphism $\varphi_1: A_1 \to A_1^\wedge$ and an elementary embedding $\delta^\wedge : A^\wedge \to A_1^\wedge$ making the diagram

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\delta"'] & A^\wedge \arrow[d, "\delta^\wedge"] \\
A_1 \arrow[r, "\varphi_1"] & A_1^\wedge
\end{tikzcd}$$

commutative. Similarly, the automorphism group $\mathrm{Aut}(\boldsymbol{A})$ acts on $\boldsymbol{A}_1$ by $\delta:\boldsymbol{A}\to\boldsymbol{A}_1$,

$$\begin{array}{ccc} \boldsymbol{A} & \xrightarrow{\varphi} & \boldsymbol{A} \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta^\wedge} \\ \boldsymbol{A}_1^\wedge & \xrightarrow{\varphi_1} & \boldsymbol{A}_1^\wedge \end{array}$$

is commutative and $\varphi\mapsto\varphi_1$ is a group homomorphism.

THEOREM. *$\boldsymbol{A}^*$ is elementarily lifted over $\boldsymbol{A}$ iff there is a diagonal-like embedding $\delta:\boldsymbol{A}\to\boldsymbol{A}^*$.*

Proof. Assume first $\boldsymbol{A}<\boldsymbol{A}^*$. Let Φ be the set of all elementary maps $\varphi:\boldsymbol{A}\to\boldsymbol{A}^*$ which are not diagonal-like. For every $\varphi\in\Phi$ we have an axiomatic class $\mathfrak{B}_\varphi$, an axiomatic relation Γ_φ between $\mathfrak{A}_\varrho$ and $\mathfrak{B}_\varphi$, described by K_φ, some $\boldsymbol{B}_\varphi\in\mathfrak{B}_\varphi$ and an interpretation $\mathscr{I}_\varphi$ of the relations in Θ_φ on $\boldsymbol{A}\dot\cup\boldsymbol{B}_\varphi$ such that

$$(\boldsymbol{A}\dot\cup\boldsymbol{B}_\varphi)^\wedge \Vdash K_\varphi$$

holds, but there is no extension of φ to an elementary map from $(\boldsymbol{A}\dot\cup\boldsymbol{B}_\varphi)^\wedge$ into $(\boldsymbol{A}^*\dot\cup\boldsymbol{B}_\varphi^*)^\wedge$, no matter how $\boldsymbol{B}_\varphi^*\in\mathfrak{B}_\varphi$ is chosen and no matter how Θ_φ is interpreted on $\boldsymbol{A}^*\dot\cup\boldsymbol{B}_\varphi^*$.

We form the "direct sum" $\boldsymbol{C}=\big(\boldsymbol{A}\dot\cup(\boldsymbol{B}_\varphi)_{\varphi\in\Phi}\big)^\wedge$. Besides the fundamental relations on $\boldsymbol{A}$ and on every $\boldsymbol{B}_\varphi$ we have on $\boldsymbol{C}$ unary relations S_A and S_φ designating the pairwise disjoint sets A and B_φ, $\varphi\in\Phi$. On every $\boldsymbol{A}\dot\cup\boldsymbol{B}_\varphi$ the relations in Θ_φ are interpreted according to $\mathscr{I}_\varphi$. Now, let $\mathfrak{B}$ be the axiomatic class of all elementary extensions of $\boldsymbol{C}$ and let K describe the axiomatic relation Γ given by: $\boldsymbol{A}'\in\mathfrak{A}_\varrho$ is ϱ-isomorphic to the $\mathfrak{A}$-component of a $\boldsymbol{C}'\in\mathfrak{B}$. We have $(\boldsymbol{A}\dot\cup\boldsymbol{C})^\wedge\Vdash K$ and in spite of $\boldsymbol{A}<\boldsymbol{A}^*$ we get $(\boldsymbol{A}^*\dot\cup\boldsymbol{C}^*)^\wedge\Vdash K$ with a suitable $\boldsymbol{C}^*\in\mathfrak{B}$. Identification of $\boldsymbol{A}^*$ with the $\mathfrak{A}$-component in $\boldsymbol{C}^*$ yields an elementary embedding

$$\delta^\wedge:\big(\boldsymbol{A}\dot\cup(\boldsymbol{B}_\varphi)_{\varphi\in\Phi}\big)^\wedge\to\big(\boldsymbol{A}^*\dot\cup(\boldsymbol{B}_\varphi^*)_{\varphi\in\Phi}\dot\cup S\big)^\wedge,$$

where S is a discrete "rest". The restriction

$$\delta=\delta^\wedge_{|A}:\boldsymbol{A}\to\boldsymbol{A}^*$$

is elementary as are all of

$$\delta^\wedge_{|A\dot\cup B_\varphi}:(\boldsymbol{A}\dot\cup\boldsymbol{B}_\varphi)^\wedge\to(\boldsymbol{A}^*\dot\cup\boldsymbol{B}_\varphi^*)^\wedge,\qquad \varphi\in\Phi.$$

Therefore, $\delta\notin\Phi$, δ is diagonal-like.

The converse if obvious.

As a straight-forward application of Vaught-Tarski theorem about elementary chains we observe

Corollary 1. *Let A be the direct limit of an ascending chain $(A_\nu)_{\nu<\lambda}$ of structures with diagonal-like embeddings $\delta_{\nu\mu}: A_\nu \to A_\mu$, $\nu < \mu$. Then the limit embeddings $\delta_\nu: A_\nu \to A$ are diagonal-like, $A_\nu < A$ for all $\nu < \lambda$.*

This corollary generalizes the fact that any projective class is closed under ultralimits.

Corollary 2. *Let $\delta_i: A \to A_i$, $i \in I$, be diagonal-like and let* $\mathfrak{u}$ *be an ultrafilter on I. Then*

$$\delta: A \to A^I/\mathfrak{u} \xrightarrow{\times_i \delta_i/\mathfrak{u}} \mathop{\times}_{i \in I} A_i/\mathfrak{u}$$

is diagonal-like.

For certain axiomatic classes $\mathfrak{A}$, any elementary extension A' of an $A \in \mathfrak{A}$ is an elementarily lifted extension over A. The next proposition has the same proof as a well-known two cardinal theorem (cf. Bell and Slomson [4], p. 247).

Proposition (G. C. H.). *Let $\mathfrak{A}$ be categorical for all cardinals not less than* $\mathfrak{m}$ *and let $A, A' \in \mathfrak{A}$. Then if* $\operatorname{card}(A) = \alpha \geqslant \mathfrak{m} \geqslant \aleph_0$ *and* $\operatorname{card}(A') = \alpha' \geqslant \alpha$, *we have $A < A'$.*

We assume the G. C. H. to assure for any cardinal $\alpha' \geqslant \alpha$ an iterated ultralimit over A of this cardinal.

Under the hypothesis of the proposition the intersection of all projective classes containing A is quasi-axiomatic (cf. Grätzer [5], p. 259). The algebraically closed field C of complex numbers gives an example.

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