COLLOQUIUM MATHEMATICUM

VOL. XX 1969 FASC. 1

ON ACYCLIC KERNELS AND THE BARYCENTRIC HOMOMORPHISM

BY

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1. Introduction. Throughout this paper (1) we shall assume the reader is familiar with the definitions and basic facts about chain complexes, chain mappings, the homology of a chain complex, and the singular homology of a topological space. We shall use the definitions and notation of [1]. The singular chain complex of a topological space X will be denoted by $C(X)$ and the barycentric homomorphism by β. If $f: C \to C$ is a chain mapping of the chain complex C into itself and K denotes that subcomplex of C which is the kernel of f, then f is said to have an *acyclic kernel* if and only if K is an acyclic chain complex, i.e., $H_p(K) = 0$ for each p.

In [2] Fadell has shown that the kernel of the barycentric homomorphism is acyclic. This condition is not implied by the fact that β_* (the homomorphism that β induces on $H(C(X))$) is an isomorphism nor even by the fact that $B \sim 1$ (β is chain homotopic to the identity homomorphism). It is implied by the fact that there is, as Fadell shows, a chain homotopy connecting β and 1 which is stable with respect to the kernel of β. Precisely, there is a homomorphism $\varrho: C(X) \to C(X)$ such that (1) ϱ has degree 1, (2) $\varrho\partial + \partial\varrho = 1 - \beta$, and (3) $\varrho(K) \subset K$, where K is the kernel. When the condition given in (3) holds we say that ϱ is *stable* with respect to the kernel of β. Later we shall consider this notion in a more general setting.

The result that the kernel of β is acyclic is used in [2] to obtain an "unessential identifier" for $C(X)$. For any chain complex C and sub-

(1) Many of the results in this paper are contained in a thesis submitted to the faculty of the University of Wisconsin in partial fulfillment of the requirements for the Ph. D. degree, August 1962.

The author is grateful to Professor Edward Fadell for his encouragement and many valuable suggestions.

A part of the research for this paper was supported by NSF-GP-7189.

complex K it is shown in [3] that K is acyclic if and only if π_* is an isomorphism where $\pi : C \to C/K$ is the natural projection of C onto the factor complex C/K. Thus, an unessential identifier for C is a subcomplex which can be factored out without affecting the homology. It is not hard to show that the kernels of the iterates of β, i.e., $\beta^2 = \beta \circ \beta$, $\beta^3 = \beta \circ \beta \circ \beta$, etc., form a properly increasing sequence of subcomplexes of $C(X)$. We show in § 3 that they are all acyclic. Thus their union can be used to obtain an unessential identifier for $C(X)$ which is larger than that given in [2].

In § 2 we give some examples and facts about acyclic kernels for arbitrary chain complexes.

2. Facts and examples about acyclic kernels. The following theorem gives a condition equivalent to the kernel being acyclic for a chain map $f: C \to C$ such that f_* is an isomorphism:

THEOREM 2.1. *If $C = \{C_p, \partial\}$ is a chain complex, $f: C \to C$ is a chain mapping, $K = \{K_p, \partial\}$ is the kernel of f, and $i : f(C) \to C$ is the inclusion homomorphism, then any two of the following conditions imply the third*:

Condition 1. *i_* is an isomorphism.*

Condition 2. *K is acyclic.*

Condition 3. *f_* is an isomorphism.*

Proof. Let $\bar{f}$: $C_p/K_p \to f(C_p)$ be defined by $\bar{f}([x]) = f(x)$. Now the theorem immediately follows from the commutativity of the following diagram and the fact, mentioned earlier, that K is acyclic if and only if π_* is an isomorphism:

$$\begin{tikzcd} H_p(C/K) \arrow[r, "\bar{f}_*"] & H_p(f(C)) \arrow[r, "i_*"] & H_p(C) \\ H_p(C) \arrow[u, "\pi_*"] \arrow[urr] & & \end{tikzcd}$$

Some examples illustrate the relationships between the conditions given in Theorem 3.2. In [3] Radó gives an example which illustrates that K may fail to be acyclic even when $f \sim 1$ (and thus $f_* = 1$). Hence, Condition 3 does not imply Condition 2. That Condition 2 does not imply Condition 3 is demonstrated by the following example.

Example 2.2. Let a chain complex $C = \{C_p, \partial\}$ be defined as follows:

$C_p = 0$ for $p \geqslant 2$ and for $p < 0$.

$C_1 = ((t))$, the free abelian group with one generator, t.

$C_0 = ((z))$, the free abelian group with one generator, z.

$\partial : C \to C$ is given by $\partial(t) = 2z$ and $\partial(C_p) = 0$ whenever $p \neq 1$.

A chain mapping $f: C \to C$ is given by the relations $f(t) = 2t$ and $f(z) = 2z$.

Note that if K_p is the kernel of $f|C_p$, then $K_p = 0$ for every p and, hence, $K = \{K_p, \partial\}$ is acyclic. However, $H_0(C)$ contains an element $[z] \neq 0$ while $f_*([z]) = [f(z)] = [2z] = 0$. Thus f is not an isomorphism.

Example 2.3 shows that Condition 1 does not imply Condition 3.

Example 2.3. Let $C_1 = \big((a_1, a_2, a_3, \ldots)\big)$, the free abelian group generated by an infinite countable set, and $C_p = 0$ for $p \neq 1$. Let the boundary operator be defined by $\partial(x) = 0$ for all x. Define a chain map $f: C \to C$ so that $f(a_1) = f(a_2) = a_1$ and $f(a_j) = a_{j-1}$ for $j \geqslant 3$. In this case $H_p(C) = C_p$ for each p. Furthermore, i_* is an isomorphism but f_* is not 1-1.

A chain complex C is said to be *free* provided that C_p is a free abelian group for each p. It is well known ([2]) that if $f: C \to C$, C is free, and f_* is an isomorphism, then f is a chain equivalence. Using this fact and Theorem 2.1 we obtain the following corollary about the chain complex $C \otimes G$ obtained from tensoring a free chain complex C with an abelian group G. For $f: C \to C, f \otimes 1$ denotes that chain map of $C \otimes G$ into itself defined by $f(x \otimes g) = f(x) \otimes g$.

Corollary 2.4. *Suppose that $f: C \to C$ is a chain map of a free chain complex C into itself, f_* is an isomorphism, and f has an acyclic kernel. Then the chain map $f \otimes 1: C \otimes G \to C \otimes G$ has an acyclic kernel.*

Proof. By Theorem 2.1, i_* is an isomorphism where $i: f(C) \to C$. Since both of C and $f(C)$ are free chain complexes, f and i are both chain equivalences. It follows that $(f \otimes 1)_*$ and $(i \otimes 1)_*$ are both isomorphisms. Note that here $i \otimes 1: f(C) \otimes G \to C \otimes G$. However, $f(C) \otimes G = (f \oplus 1)(C \otimes G)$, if we consider $f(C) \otimes G$ as a subgroup of $C \otimes G$. Now, applying Theorem 2.1 to the chain complex $C \otimes G$, we conclude that the kernel of $f \otimes 1$ is acyclic.

Next we give an example which deals with the iterates of a chain mapping. Example 2.5 shows that for every integer $n \geqslant 1$ there is a chain complex C and a chain mapping $f: C \to C$ such that $f \sim 1$ (and thus $f^k \sim 1$ for each k), the kernel of f^k is acyclic for $k \neq n$, but the kernel of f^n is not acyclic.

Example 2.5. Let n be a fixed integer greater than 1. (An easy modification works for $n = 1$.)

Define a chain complex $C = \{C_p, \partial\}$ as follows:

$C_p = 0$ for $p > 2$ and $p < 0$.

$C_2 = \big((s_1, s_2, \ldots, s_n)\big)$, the free abelian group with the n generators $s_1, s_2, \ldots, s_n$.

([2]) For example, see p. 192 of [5].

$C_1 = \big((a_0, a_1, \ldots, a_n, b_1, b_2, \ldots, b_{n-1})\big)$, the free abelian group with the $2n$ generators shown.

$C_0 = \big((x_1, x_2, \ldots, x_n)\big)$, the free abelian group with the n generators shown.

$\partial : C \to C$ is given by

$$\begin{aligned}
\partial(C_p) &= 0 \quad \text{for } p > 2 \text{ and } p < 1,\\
\partial(s_1) &= a_0,\\
\partial(s_j) &= a_{j-1} - b_{j-1} \quad \text{for } 2 \leqslant j \leqslant n,\\
\partial(a_0) &= 0,\\
\partial(a_j) &= x_j \quad \text{for } i \leqslant j \leqslant n,\\
\partial(b_j) &= x_j \quad \text{for } 1 \leqslant j \leqslant n-1.
\end{aligned}$$

Define a homomorphism $f : C \to C$ as follows:

$f(C_p) = 0$ for $p > 2$ and $p < 0$.

$f | C_2$ is given by $f(s_1) = 0$ and $f(s_j) = s_{j-1}$ for $2 \leqslant j \leqslant n$.

$f | C_1$ is given by $f(a_0) = f(b_1) = 0$, $f(a_j) = a_{j-1}$ for $1 \leqslant j \leqslant n$ and $f(b_j) = b_{j-1}$ for $2 \leqslant j \leqslant n-1$.

$f | C_0$ is given by $f(x_1) = 0$ and $f(x_j) = x_{j-1}$ for $2 \leqslant j \leqslant n$.

It is easy to show that f is a chain mapping.

Define a homomorphism $\varrho : C \to C$ as follows:

$\varrho(C_p) = 0$ for $p > 1$ and $p < 0$.

$\varrho | C_0$ is given by $\varrho(x_j) = b_j - a_{j-1}$ for $1 \leqslant j \leqslant n-1$ and $\varrho(x_n) = a_n - a_{n-1}$.

$\varrho | C_1$ is given by $\varrho(a_j) = s_{j+1}$ for $0 \leqslant j \leqslant n-1$, $\varrho(a_n) = 0$ and $\varrho(b_j) = s_j$ for $1 \leqslant j \leqslant n-1$.

It can be verified that $\varrho\partial + \partial\varrho = 1 - f$. Hence $f \sim 1$.

Hereafter, $\big((y_1, y_2, \ldots, y_m)\big)$ will be used as above to denote the free abelian group with the m generators $y_1, y_2, \ldots, y_m$. Also $K^j = \{K^j_p, \partial\}$ will denote the kernel of f^j.

A consideration of the definition of f shows that the kernels are as follows:

$K^j_p = 0$ for all j whenever $p < 0$ or $p > 2$.

$K^j_2 = \big((s_1, s_2, \ldots, s_j)\big)$ for $1 \leqslant j \leqslant n-1$ and $K^j_2 = C_2$ for $j \geqslant n$.

$K^j_1 = \big((a_0, \ldots, a_{j-1}, b_1, \ldots, b_j)\big)$ for $1 \leqslant j \leqslant n-1$, $K^n_1 = \big((a_0, \ldots, a_{n-1}, b_1, \ldots, b_{n-1})\big)$, and $K^j_1 = C_1$ for $j \geqslant n+1$.

$K^j_0 = \big((x_1, \ldots, x_j)\big)$ for $1 \leqslant j \leqslant n-1$ and $K^j_0 = C_0$ for $j \geqslant n$.

Since $K^j_p = 0$ for all j whenever $p > 2$ or $p < 0$, $H_p(K^j) = 0$ for all j whenever $p > 2$ or $p < 0$.

Since C_2 contains no cycles, $H_2(K^j) = 0$ for all j.

For the case $p = 1$, $K_1^1 \cap Z_1 = \big((a_0)\big)$,

$$K_1^j \cap Z_1 = \big((a_0, a_1 - b_1, a_2 - b_2, \ldots, a_{j-1} - b_{j-1})\big) \quad \text{for } 2 \leqslant j \leqslant n,$$

and

$$K_1^j \cap Z_1 = \big((a_0, a_1 - b_1, a_2 - b_2, \ldots, a_{n-1} - b_{n-1})\big) \quad \text{for } j \geqslant n.$$

Consideration of the definition of ∂ shows that $\partial(K_2^j) = K_1^j \cap Z_1$ for every j. This implies that $H_1(K^j) = 0$ for every j.

For the case $p = 0$, $C_0 = Z_0$ and thus $K_0^j \cap Z_0 = K_0^j$. Consideration of the definition of ∂ shows that $\partial(K_1^j) = K_0^j$ whenever $j \neq n$. Hence $H_0(K^j) = 0$ for all $j \neq n$. However, $x_n \in K_0^n \cap Z_0$ and $x_n \notin \partial(K_1^n)$ so $H_0(K^n) \neq 0$.

3. The kernels of the iterates of β are acyclic. We start by giving a brief discussion of the notation which we shall use. Let E_∞ denote the set of all square summable real sequences with the usual topology. Let d_0, $d_1, d_2, \ldots$ denote, respectively, the points $(1, 0, 0, 0, \ldots)$, $(0, 1, 0, 0, \ldots)$, $(0, 0, 1, 0, \ldots)$, $\ldots$ in E_∞. If $v_0, v_1, v_2, \ldots, v_p$ are any $p+1$ points in E, then $\langle v_0 v_1 \ldots v_p \rangle$ denotes the convex hull of these points. We set $\Delta_p = \langle d_0 d_1 \ldots d_p \rangle$. For any topological space X, $C_p(X)$ is the free abelian group generated by all continuous mappings $T : \Delta_p \to X$. For any $p+1$ points $v_0, v_1, \ldots, v_p$ in E_∞, $[v_0 v_1 \ldots v_p]$ will stand for that linear map L of Δ_p into E_∞ with the property that $L(d_i) = v_i$ for each i and $b(v_0, v_1, \ldots, v_p)$ will denote the barycenter of the $p+1$ points, i.e., the point $(v_0 + v_1 + \ldots + v_p)/p+1$. We shall assume familiarity with the barycentric homomorphism $\beta : C(X) \to C(X)$. In any case it suffices, for our purposes, to state that for $T : \Delta_p \to X$, $\beta(T) = \sum_{\sigma \epsilon Pn} \operatorname{sgn} \sigma \, (T \circ [\sigma])$, where P_n is the set of all permutations of the set $\{0, 1, 2, \ldots, n\}$ and for $\sigma = (i_0, i_1, \ldots, i_n)$ ϵP_n, $[\sigma] = [d_{i_0}, b(d_{i_0} d_{i_1}), b(d_{i_0} d_{i_1} d_{i_2}), \ldots, b(d_{i_0} d_{i_1} \ldots d_{i_n})]$ and $\operatorname{sgn} \sigma$ is 1 when σ is even, and -1 when σ is odd.

Straightforward computation yields the following useful lemma.

LEMMA 3.1. *If X is a topological space, T is a map of Δ_p into X, and L is a linear map of Δ_n into Δ_p such that $L(d_j) = L(d_k)$ where d_j and d_k are distinct vertices of Δ_n, then $\beta(T \circ L) = 0$.*

The following notation will be used hereafter: If $n \geqslant 0$ and $f : \Delta_n \to \Delta_n$ is a map such that $f \circ [\sigma]$ is linear for every $\sigma \epsilon P_n$, then $D(n, f) = \{\sigma \epsilon P_n \,|\, f \circ [\sigma]$ agrees on two distinct vertices of $\Delta_n\}$. Note that $D(0, f) = \emptyset$ for any map f from Δ_0 to Δ_0.

LEMMA 3.2. *If X is a topological space, T is a map of Δ_n into X with $n \geqslant 0$, and $f : \Delta_n \to \Delta_n$ is a map such that $f \circ [\sigma]$ is linear for every $\sigma \epsilon P_n$ and $J = P_n - D(n, f)$, then*

$$\beta^2(T \circ f) = \sum_{\sigma \epsilon J} \Big(\sum_{\tau \epsilon P_n} (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau)(T \circ f) \circ [\sigma] \circ [\tau] \Big).$$

Proof. Write $\beta^2(T\circ f) = \beta(\beta(T\circ f)) = \beta(A+B) = \beta(A)+\beta(B)$ where

$$A = \sum_{\sigma\epsilon J}\operatorname{sgn}\sigma\big((T\circ f)\circ[\sigma]\big) \quad \text{and} \quad B = \sum_{\sigma\epsilon D(n,f)}\operatorname{sgn}\sigma\big((T\circ f)\circ[\sigma]\big).$$

If $\sigma\epsilon D(n,f)$, then $f\circ[\sigma]$ agrees on two distinct vertices of Δ_n and thus satisfies the hypothesis of Lemma 3.1. It follows that $\beta\big((T\circ f)\circ[\sigma]\big) = 0$. Thus $\beta(B) = 0$ and

$$\begin{aligned}\beta^2(T\circ f) = \beta(A) &= \beta\Big(\sum_{\sigma\epsilon J}\operatorname{sgn}\sigma\big((T\circ f)\circ[\sigma]\big)\Big)\\ &= \sum_{\tau\epsilon P_n}\Big(\sum_{\sigma\epsilon J}(\operatorname{sgn}\tau)(\operatorname{sgn}\sigma)(T\circ f)\circ[\sigma]\circ[\tau]\Big).\end{aligned}$$

The order of the summation signs may be changed proving the lemma.

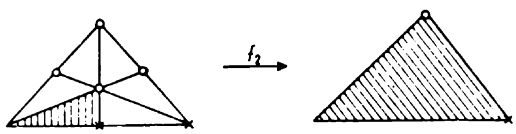
Definition of $f_n : \Delta_n \to \Delta_n$. For each integer $n \geqslant 0, f_n$ is to be a map with the following three properties:

P-1. $f_n\circ[\sigma]$ is linear for each $\sigma\epsilon P_n$, i.e., f_n is linear on the first barycentric subdivision of Δ_n.

P-2. If σ_0 is the identity permutation in P_n, then $f_n\circ[\sigma_0]$ is the identity map.

P-3. If σ is not the identity permutation in P_n, then $\sigma\epsilon D(n,f_n)$, i.e., $f_n\circ[\sigma]$ agrees on two distinct vertices of Δ_n.

The f_n are to be defined inductively. Roughly, letting σ_0 be the identity permutation in P_n, f_n is to expand the "σ_0-piece" of Δ_n onto all of Δ_n in a natural linear manner while collapsing the other pieces linearly into the boundary in such a way that for each such piece two distinct vertices are mapped into the same point. This is illustrated as follows for $n = 2$.



Precisely, we use the fact that $\Delta_1 \subset \Delta_2 \subset \Delta_3 \ldots$, set $f_0(d_0) = d_0$ and define f_n to be linear on the first barycentric subdivision of Δ_n with $f_n|\Delta_{n-1} = f_{n-1}$ and $f_n(d) = d_n$ for each d which is a vertex of the first barycentric subdivision of Δ_n and is not in Δ_{n-1}.

Properties P-1 and P-2 are easily verified from the definitions. We give an inductive proof that P-3 is satisfied.

Proof of P-3 is trivially true when $n = 0$. Assume that $P_{n-1} - D(n-1, f_{n-1})$ contains only the identity permutation. Now let $\sigma\epsilon P_n$ with $\sigma \neq$ the identity. Either the σ-piece of Δ_n has a face in Δ_{n-1} or it does not; these two cases are considered as follows.

In the first case, since σ is not the identity, there is a $\tau \epsilon P_{n-1}$ such that τ is not the identity and $[\sigma]|\Delta_{n-1} = [\tau]$. But then, by the inductive assumption, $\tau \epsilon D(n-1, f_{n-1})$ and $f_n \circ [\sigma]|\Delta_{n-1} = f_{n-1} \circ [\tau]$ agrees on two distinct vertices of Δ_{n-1}. Thus $f_n \circ [\sigma]$ agrees on two distinct vertices of n and $\sigma \epsilon D(n, f_n)$.

In the second case, $\sigma = (i_0, i_1, \ldots, i_n)$ where $i_j = n$ for some $j < n$. Hence $[\sigma](d_j) = b(d_{i_0} d_{i_1} \ldots d_{i_j}) \notin \Delta_{n-1}$ and $[\sigma](d_n) = b(d_{i_0} d_{i_1} \ldots d_{i_n}) \notin \Delta_{n-1}$. By definition $(f_n \circ [\sigma])(d_j) = (f_n \circ [\sigma])(d_n) = d_n$ and it is proved that $\sigma \epsilon D(n, f_n)$.

THEOREM 3.3. *If $C(X) = \{C_n(X), \partial\}$ is the chain complex of singular chains of a topological space X and β is the barycentric homomorphism, then $\beta(C(X)) = \beta^j(C(X))$ for every positive integer j.*

Proof. Note that $\beta(C(X)) \supset \beta^2(C(X)) \supset \beta^3(C(X)) \supset \ldots$ Therefore it remains to show that $\beta(C(X)) \subset \beta^2(C(X)) \subset \beta^3(C(X)) \subset \ldots$ or, equivalently, simply that $\beta(C(X)) \subset \beta^2(C(X))$.

Let $T \epsilon C_n(X)$ be an arbitrary generator of $C(X)$ and let $f_n\colon \Delta_n \to \Delta_n$ be the map defined above.

By Lemma 3.2,

$$\beta_2(T \circ f_n) = \sum_{\sigma \epsilon P_n - D(n, f_n)} \Big(\sum_{\tau \epsilon P_n} (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau) T \circ f_n \circ [\sigma] \circ [\tau] \Big).$$

Applying P-3,

$$\beta^2(T \circ f_n) = \sum_{\tau \epsilon P_n} \operatorname{sgn} \tau (T \circ f_n) \circ [\sigma_0] \circ [\tau],$$

where σ_0 is the identity permutation in P_n. Applying P-2, $f_n \circ [\sigma_0]$ is the identity map and

$$\beta^2(T \circ f_n) = \sum_{\tau \epsilon P_n} (\operatorname{sgn} \tau) T \circ [\tau] = \beta(T).$$

This shows that $\beta(T) \epsilon \beta^2(C(X))$ for any generator T of $C(X)$ and thus that $\beta(C(X)) \subset \beta^2(C(X))$.

THEOREM 3.4. *If $C(X)$ is the chain complex of singular chains of a topological space X, β is the barycentric homomorphism, j is a positive integer, and K^j denotes the kernel of chain mapping β^j, then K^j is acyclic.*

Proof. Consider the following diagrams, where i, i', and i'' are the inclusion chain maps:

$$\begin{tikzcd} H(\beta^j(C(X))) \arrow[rr, "i_*"] \arrow[dr, "i'_*"'] & & H(C(X)) \\ & H(\beta(C(X))) \arrow[ur, "i''_*"'] & \end{tikzcd}$$

By Theorem 3.3, i' is an isomorphism of $\beta^j(C(X))$ onto $\beta(C(X))$ and thus i'_* is an isomorphism. Also, β is an isomorphism and Fadell proved in [2] that the kernel of β is acyclic. Hence, by Theorem 2.1, i''_* is an isomorphism. Then the commutativity of the diagram yields the conclusion that i_* is an isomorphism. Applying Theorem 2.1 again gives the result that K^j, the kernel of β^j, is acyclic and completes the proof of the theorem.

4. On chain homotopies which are stable with respect to the kernel. If $f: C \to C$ and there is a chain homotopy ϱ connecting f and 1 which has the property that $\varrho(K) \subset K$, where K is the kernel of f, then it is easy to show that K is acyclic. In fact, we stated earlier that Fadell used such a homotopy to show that the kernel of β is acyclic. It might be difficult to apply this technique to β^2 since the usual chain homotopies connecting β^2 and 1 are not stable with respect to the kernel. For example, if ϱ is the chain homotopy connecting β and 1 given in [4] and $\tilde{\varrho} = \varrho + \varrho\beta$, then $\tilde{\varrho}$ is a homotopy connecting β^2 and 1 which is not stable with respect to the kernel. Stable homotopy operators for β^2 do exist, however, as the following theorem implies:

THEOREM 4.1. *If C is a free chain complex and f is a chain mapping from C to C such that $f \sim 1$ and the kernel of f is acyclic then there is a chain homotopy connecting f and 1 which is stable with respect to K, the kernel of f.*

The following known result (3) is useful in proving Theorem 4.1. We state it as a lemma.

LEMMA 4.2. *Given the hypothesis of Theorem 4.1, there is a chain mapping $g: f(C) \to C$ such that $f \circ g = 1$.*

Proof of Theorem 4.1. Z_p^K and B_p^K will denote the p-cycles and p-bounds, respectively, of K. Since B_{p-1}^K is a free abelian group for every p, there is a homomorphism r of degree 1 from ΣB_p^K to ΣK_p and, for every p, a split exact sequence $0 \to Z_p^K \xrightarrow{j} K_p \underset{r}{\overset{\partial}{\rightleftarrows}} B_{p-1}^K \to 0$, where j is the inclusion homomorphism. Furthermore, since K is acyclic, $H_p(K) = 0$ and $Z_p^K = B_p^K$ for every integer p. Thus we can write that the sequence $0 \to Z_p^K \xrightarrow{j} K_p \underset{r}{\overset{\partial}{\rightleftarrows}} Z_{p-1}^K \to 0$ is split exact. Then $K_p = Z_p^K \oplus r(Z_{p-1}^K)$ is the direct sum of two of its subgroups.

Define a homomorphism $\tilde{r}$ of degree 1 from ΣK_p to ΣK_p by considering $k = z + r(z') \in K_p$ with $z \in Z_p^K$ and $r(z') \in r(Z_{p-1}^K)$ and setting $\tilde{r}(z + r(z'))$ $= r(z) \in K_{p+1}$.

(3) See exercise on p. 158-9 of [1].

Note that $\tilde{r}\,|\,\Sigma Z_p^K = r$ and $\tilde{r}\circ r = 0$ and recall that $\partial\circ r = 1$. It follows that for $z \in Z_p^K$ and $z' \in Z_{p-1}^K$, $\tilde{r}(\partial(z)) = r(0) = 0$, $\tilde{r}\big(\partial(r(z'))\big) = \tilde{r}(z') = r(z')$, $\partial(\tilde{r}(z)) = \partial(r(z)) = z$, and $\partial\big(\tilde{r}(r(z'))\big) = \partial(0) = 0$. Combining these results yields $(\tilde{r}\partial + \partial\tilde{r})(z + r(z')) = r(z') + z$. Thus $\tilde{r}\partial + \partial r = 1$.

From the hypothesis that $f \sim 1$ there is a homomorphism $\tilde{\varrho}$ of degree 1 from ΣC_p to ΣC_p such that $\tilde{\varrho}\partial + \partial\tilde{\varrho} = 1 - f$.

By Lemma 4.2, there is a chain mapping g from $f(C)$ to C such that the sequence $0 \to K_p \xrightarrow{j} C_p \underset{g}{\overset{f}{\rightleftarrows}} f(C_p) \to 0$ is split exact, where j is the inclusion homomorphism. Thus, for each p, $C_p = K_p \oplus g(f(C_p))$ is the direct sum of two of its subgroups.

Now define a homomorphism ϱ of degree 1 from ΣC_p to ΣC_p by considering $x = k + y \in C_p$ with $k \in K_p$ and $y \in g(f(C_p))$ and setting $\varrho(x) = \tilde{r}(k) + \tilde{\varrho}(y)$, where $\tilde{r}$ and $\tilde{\varrho}$ are as given above. It is easy to show that $\varrho\partial + \partial\varrho = 1 - f$. Also, $\varrho(K_p) = \tilde{r}(K_p) \subset K_{p+1}$, so ϱ is stable with respect to the kernel of f and the theorem is proved.

5. Remarks and questions. Corollary 2.4 shows that acyclicity of the kernel of $f \otimes 1$ follows from that of f in the case where C is free and f_* is an isomorphism. The proof made no use of any relationship between $\mathrm{Ker}(f\otimes 1)$ and $(\mathrm{Ker} f)\otimes G$. If $\mathrm{Ker}(f\otimes 1) = (\mathrm{Ker} f)\otimes G$, then one could use the stable homotopy operator given by Theorem 4.1 to obtain an alternate proof for the corollary. Is it in fact true that the equality just mentioned must hold given the hypothesis of Corollary 2.4? (**P 651**)

Results in [2], [3], and [4] are concerned with finding unessential identifiers for the complex $C(X)$ and for a somewhat larger complex $R(X)$. In [2] Fadell obtains a largest known unessential identifier for $R(X)$. A still larger one could be obtained using our result on the acyclicity of the kernels of the iterates of β. Is there an even larger one (**P 652**)? Or, we can ask related questions concerning $C(X)$. First let $K^j(X)$ denote the kernel of β^j. Let $K(X) = \bigcup_{j=1}^{\infty} K^j(X)$. It is easy to see that $K(X)$ is an unessential identifier for $C(X)$. Is there a larger one? Is there a largest unessential identifier for $C(X)$? (**P 653**)

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Reçu par la Rédaction le 16. 10. 1967