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SPACES FOR WHICH THE DIAGONAL HAS A CLOSED NEIGHBORHOOD BASE

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0. Introduction. In this paper we study the notion of diagonal normality. A space X is called *diagonal normal* (*Δ-normal* for short) if the diagonal ΔX has a closed neighborhood base in $X \times X$. As in the case of divisible spaces (the collection of neighborhoods of ΔX forms a uniformity), Δ-normality is obtained by isolating a property of uniformities and considering it in the family of all neighborhoods of ΔX. Contrary to divisibility, which implies collectionwise normality, Δ-normality has no nice relations with the usual separation axioms: Δ-normality implies regularity but it does not imply complete regularity. It is however still possible that, for example, some nice results may be proved about spaces which are Δ-normal and normal. For example, Δ-normality features in a characterization of divisible spaces [8]. The paper is organized as follows: Section 1 contains the relevant definitions, Section 2 contains some general theorems on Δ-normality, among them a characterization in terms of open covers and Section 3 contains some examples to show that in general Δ-normality behaves rather badly. Finally the author would like to thank Eva Lowen-Colebunders for bringing the subject of Δ-normality to his attention.

1. Definitions and preparatory remarks. Since we will be working with product of sets we first fix some terminology on this subject. Let X be a set. ΔX denotes the diagonal of X, i.e. the subset $\{\langle x, x\rangle |\ x \in X\}$ of $X \times X$. If $A \subseteq X \times X$ and $x \in X$ we let $A[x] = \{y \in X|\ \langle x, y\rangle \in A\}$. Note that if X is a topological space and A is open then also $A[x]$ is open. Furthermore if $A, B \subseteq X \times X$ we put

$$A \circ B = \{\langle x, y\rangle |\ \exists z \in X\colon\ \langle z, y\rangle \in A \wedge \langle x, z\rangle \in B\}.$$

We now turn to the topological definitions. Let X be a topological space. We call X *diagonal normal* (*Δ-normal*) if ΔX has a closed neighborhood base in $X \times X$, or equivalently for all open $U \supseteq \Delta X$ there is an open $V \supseteq \Delta X$

such that $\bar{V} \subseteq U$, or, to make it sound it even more "normal", any two disjoint closed subsets of $X \times X$ one of which is the diagonal can be separated by open sets.

In Section 2 we exhibit some properties of Δ-normal spaces and in Section 3 we give some examples. For all standard topological notions we refer to [4], the non-standard ones will be given below.

Let X be a topological space; X is called

(i) *divisible* ([2]) if every open set U containing ΔX can be divided by two, i.e. there exists an open set V around ΔX such that $V \circ V \subseteq U$;

(ii) *monotonically normal* ([6]) if one can assign to all pairs $\langle A, U \rangle$ with A closed, U open and $A \subseteq U$ an open set $G(A, U)$ such that $A \subseteq G(A, U) \subseteq \overline{G(A, U)} \subseteq U$ and if $A \subseteq A'$ and $U \subseteq U'$ then $G(A, U) \subseteq G(A', U')$;

(iii) *retractable* if every closed subset A of X is a retract of X, i.e. there is a continuous map $r\colon X \to A$ such that $r(x) = x$ for all $x \in A$. See [3] for an investigation of retractability.

The above-mentioned properties are in fact quite strong separation properties, for instance they all imply collectionwise normality (spaces with properties (ii) or (iii) are even hereditarily CWN).

Finally $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ denote the sets of irrational, rational and real numbers respectively.

2. Some positive results. In this section we characterize Δ-normality in terms of open covers of X, next we indicate the position of Δ-normality among other separation axioms and finally we prove a theorem on normality of Δ-normal spaces. The characterization of Δ-normality which we are about to give is not particularly nice, but it is sometimes easier to work with than the original definition (see the examples).

2.1. THEOREM. *Let X be a topological space. Then X is Δ-normal iff every open cover $\mathscr{U}$ of X has an open refinement $\mathscr{V}$ with the following property*: $\forall x, y \in X$: *if* $\forall U \in \mathscr{U}$: $x \notin U \vee y \notin U$, *then there are open $O_x \ni x$ and $O_y \ni y$ such that* $\forall V \in \mathscr{V}$: $O_x \cap V = \emptyset \vee O_y \cap V = \emptyset$.

Proof. "$\Rightarrow$". Given $\mathscr{U}$, let $U = \bigcup \{O \times O \mid O \in \mathscr{U}\}$. Take $V \supseteq \Delta X$ open such that $\bar{V} \subseteq U$, and let $\mathscr{V} = \{W \subseteq X \mid W$ is open, $W \subseteq$ some $U \in \mathscr{U}$, $W \times W \subseteq V\}$.

"$\Leftarrow$". Given U, let $\mathscr{U} = \{O \subseteq X \mid O$ open and $O \times O \subseteq U\}$. Take a refinement $\mathscr{V}$ of $\mathscr{U}$ having the property described above, and let $V = \bigcup \{W \times W \mid W \in \mathscr{V}\}$.

The following theorem puts Δ-normality in relation with other separation and covering axioms.

2.2. THEOREM. (a) *Divisible spaces are Δ-normal*;

(b) *Δ-normal T_1-spaces are regular.*

Proof. (a) Let X be divisible. Let $U \supseteq \Delta X$ be open and take $V \supseteq \Delta X$ open such that $V \circ V \circ V \subseteq U$. It is easy to see that $\bar{V} \subseteq V \circ V \circ V$, so we have $\bar{V} \subseteq U$.

(b) Let X be Δ-normal and T_1. Let $A \subseteq X$ be closed and let $x \in X \setminus A$. $\{x\} \times A$ is closed in $X \times X$ and disjoint from ΔX. So let $V \supseteq \Delta X$ be open such that $\bar{V} \cap (\{x\} \times A) = \emptyset$. Then $V[x]$ is a neighborhood of x and $\overline{V[x]} \cap A = \emptyset$. $\square$

From 2.2 (a) it follows that a large class of spaces is Δ-normal, for example paracompact spaces are Δ-normal, because they are known to be divisible [7]. However in Section 3 we shall give an example of a space which is "supernormal" (meaning that it has some strong separation properties) which is not Δ-normal, and an example of a Δ-normal space which is not completely regular, so Theorem 2.2 is best possible.

The next lemma is handy in showing that some spaces are not Δ-normal, moreover it has an interesting corollary.

2.3. LEMMA. *Let X be a Δ-normal space. Let A and B be closed and disjoint subsets of X such that A is countably compact and X is Fréchet at the points of B. Then A and B can be separated by disjoint open sets.*

Proof. $A \times B$ is closed in $X \times X$ and disjoint from ΔX. So let $V \supseteq \Delta X$ be open such that $\bar{V} \cap (A \times B) = \emptyset$. Let $O = \bigcup_{a \in A} V[a]$. O is open and contains A. We show that $\bar{O} \cap B = \emptyset$. Suppose $\exists\, b \in \bar{O} \cap B$. X is Fréchet at b, so we can find a sequence $\langle x_i \rangle_{i \in \omega}$ in O such that $x_i \to b$. For all $i \in \omega$ pick $a_i \in A$ such that $x_i \in V[a_i]$. The set $\{a_i\}_{i \in \omega}$ is infinite since $B \cap \overline{V[a]} = \emptyset$ for all $a \in A$. Let $a \in A$ be an accumulation point of $\{a_i\}_{i \in \omega}$. Then, since for all i $\langle a_i, x_i \rangle \in V$,

$$\langle a, b \rangle \in (A \times B) \cap \overline{\{\langle a_i, x_i \rangle\}_{i \in \omega}} \subseteq (A \times B) \cap \bar{V},$$

which is a contradiction. $\square$

2.4. COROLLARY. *A Δ-normal space which is countably compact and Fréchet (or more special: first-countable) is normal.* $\square$

This shows that Δ-normality is stronger than regularity, since regular, countably compact, first-countable, non-normal spaces exist [12]. Our last result in this section deals with invariance of Δ-normality under mappings.

2.5. THEOREM. *Let f: $X \to Y$ be a perfect open map. Then Y is Δ-normal if X is Δ-normal.*

Proof. The map $f \times f$: $X \times X \to Y \times Y$ is also perfect and open. Let $U \supseteq \Delta Y$ be open. Take $V \supseteq \Delta X$ open such that $\bar{V} \subseteq (f \times f)^{\leftarrow}[U]$. Since $f \times f$ is open $f \times f[V]$ is a neighborhood of ΔY and since $f \times f$ is closed we have $\overline{f \times f[V]} = f \times f[\bar{V}] \subseteq U$. $\square$

In Section 3 we shall give an example of a closed map each fiber of which has at most two points, which does not preserve Δ-normality.

3. Examples. In this section we give some examples to show that Theorem 2.2 is sharp and that Δ-normality behaves badly under topological operations.

3.1. Example. A monotonically normal, hereditarily countably paracompact space which is not Δ-normal.

This is the "supernormal" space mentioned in Section 2. The above-mentioned separation properties are very strong, spaces with these properties have very nice properties with respect to extension of continuous functions [3]. Let X be the space obtained from the product $\omega_1 \times (\omega_1 + 1)$ by making each point $\langle \alpha, \beta \rangle$ with $\beta < \omega_1$ isolated. H. E. Cohen [1] showed that this space which he attributed to R. H. Bing is collectionwise normal but not divisible. E. K. van Douwen [3] showed that it has the other above-mentioned properties.

We show that X is not Δ-normal.

Let $\mathscr{U} = \{[0, \alpha] \times (\alpha, \omega_1]\}_{\alpha \in \omega_1} \cup \{\omega_1 \times \omega_1\}$. Let $\mathscr{V}$ be an open refinement of $\mathscr{U}$. For every $\alpha \in \omega_1$ pick $\gamma_\alpha < \alpha$ and $\beta_\alpha \in \omega_1$ such that

$$\langle \alpha, \omega_1 \rangle \in (\gamma_\alpha, \alpha] \times (\beta_\alpha, \omega_1] \subseteq \text{some } V_\alpha \in \mathscr{V}.$$

By the Pressing Down Lemma let $S \subseteq \omega$ and $\gamma \in \omega_1$ be such that S is unbounded and for all $\alpha \in S$ $\gamma_\alpha = \gamma$. The set $C = \{\xi \mid \alpha < \xi \Rightarrow \beta_\alpha < \xi\}$ is c.u.b. in ω_1. Let $\alpha \in C$ be a limit point $-$in ω_1- of S. Let $x = \langle \gamma + 1, \alpha \rangle$ and $y = \langle \alpha, \omega_1 \rangle$. Then $x \in \omega_1 \times \omega_1$ but $y \notin \omega_1 \times \omega_1$ and if $x \in [0, \beta] \times (\beta, \omega_1]$ then $\beta < \alpha$ so $y \notin [0, \beta] \times (\beta, \omega_1]$. Let $O_x = \{x\}$ and let $O_y = (\delta, \alpha] \times (\varepsilon, \omega_1]$ be a neighborhood of y. Take $\eta \in S \cap (\delta, \alpha)$. Then $\eta < \alpha$ so $\beta_\eta < \alpha$ and hence $x \in (\gamma, \eta] \times (\beta_\eta, \omega_1]$, and $\langle \eta, \omega_1 \rangle \in O_y$ so $O_y \cap (\gamma, \eta] \times (\beta_\eta, \omega_1] \neq \emptyset$. So for all $U \in \mathscr{U}$: $x \notin U \vee y \notin U$, but for every O_x and every O_y

$$\exists V \in \mathscr{V}: O_x \cap V \neq \emptyset \wedge O_y \cap V \neq \emptyset. \ \square$$

3.2. Example. A Δ-normal space which is not completely regular. Let X be the closed upper half-plane plus a point ∞. Points above the x-axis will be isolated.

The n-th neighborhood of $\langle x, 0 \rangle$ will be

$$\mathscr{U}_n(x) = \{\langle x, y \rangle \mid 0 \leqslant y < 1/n\} \cup \{\langle x+1+y, y \rangle \mid 0 < y < 1/n\}.$$

The n-th neighborhood of ∞ will be

$$\mathscr{U}_n(\infty) = \{\infty\} \cup \{\langle x, y \rangle \mid x > n\}.$$

A. Mysior [10] constructed this space as an example of a regular not completely regular space (a Baire-category type argument will show that the point ∞ and the closed set $\{\langle x, 0\rangle | \ x \leqslant 0\}$ cannot be separated by a continuous function).

We shall show that this space is Δ-normal.

Let $\mathcal{U} = \{\{\langle x, y\rangle\} | \ y > 0\} \cup \{U_{n_a}(a) | \ a \in \boldsymbol{R}\} \cup \{U_n(\infty)\}$ be a basic open cover of X. We shall show that $\mathcal{V} = (\mathcal{U} \setminus \{U_n(\infty)\}) \cup \{U_{n+5}(\infty)\}$ is a refinement of $\mathcal{U}$ as required by Theorem 2.2.

Let $x, y \in X$ satisfy $\forall U \in \mathcal{U}$: $x \notin U \vee y \notin U$.

(i) x is isolated. Let $O_x = \{x\}$. x can be in at most four elements of $\mathcal{V}$: $\{x\}$, two sets of the form $U_{n_a}(a)$ and $U_{n+5}(\infty)$. y is not in the first three sets which are clopen so we can choose O_y such that O_y meets none of them. If $x \in U_{n+5}(\infty)$ then $y \notin U_n(\infty) \supseteq \overline{U_{n+5}(\infty)}$ so in that case we can also require that $O_y \cap U_{n+5}(\infty) = \emptyset$. Consequently $\forall V \in \mathcal{V}$: $O_x \cap V = \emptyset \vee O_y \cap V = \emptyset$.

(ii) $x = \langle a, 0\rangle$ and $y = \infty$. In this case $x \notin U_n(\infty)$ so $a \leqslant n$ and if $U_{n_b}(b) \cap U_1(a) \neq \emptyset$ then $b < a+2 \leqslant n+3$ and so $U_{n_b}(b) \subseteq (-\infty, n+5) \times [0, \infty)$. And hence $U_{n_b}(b) \cap U_{n+5}(\infty) = \emptyset$. Consequently

$$\forall V \in \mathcal{V}: \ V \cap U_1(a) \neq \emptyset \Rightarrow V \cap U_{n+5}(\infty) = \emptyset.$$

(iii) $x = \langle a, 0\rangle$ and $y = \langle b, 0\rangle$ with $a < b$. We show that there is an $m \in \boldsymbol{N}$ such that $\forall c \in \boldsymbol{R}$: $U_m(a) \cap U_1(c) \neq \emptyset \Rightarrow U_m(b) \cap U_1(c) = \emptyset$. This immediately shows that for all $U \in \mathcal{U} \setminus \{U_n(\infty)\}$:

$$U_m(a) \cap U = \emptyset \ \vee \ U_m(b) \cap U = \emptyset.$$

Furthermore if $U_m(b) \cap U_{n+5}(\infty) \neq \emptyset$ then $b \in U_n(\infty)$ so $a \notin U_n(\infty)$ and hence $U_n(a) \cap U_{n+5}(\infty) = \emptyset$. So again $\forall V \in \mathcal{V}$:

$$U_m(a) \cap V = \emptyset \ \vee \ U_m(b) \cap V = \emptyset.$$

Also note that the reasoning in (i) and (iii) shows that $Y = X \setminus \{\infty\}$ is Δ-normal, in fact it is shown that ΔY has a clopen neighborhood base in $Y \times Y$. We shall use Y again in Example 3.3. We now turn to the finding of m. First of all note that if $U_1(c) \cap U_m(a) \neq \emptyset$ then $c \in (a-1-1/n, a-1) \cup \{a\} \cup (a+1, a+1+1/n)$ and the same is true for b of course. So we have to find $m \in \boldsymbol{N}$ such that

$$\begin{aligned}&((a-1-1/m, a-1) \cup \{a\} \cup (a+1, a+1+1/m)) \cap \\ &\qquad \cap ((b-1-1/m, b-1) \cup \{b\} \cup (b+1, b+1+1/m)) = \emptyset.\end{aligned}$$

But this is easy, just choose m such that

$$\begin{aligned}1/m &< b-a && \text{if } a < b \leqslant a+1,\\ 1/m &< b-(a+1) && \text{if } a+1 < b \leqslant a+2,\\ 1/m &< b-(a+2) && \text{if } a+2 < b. \ \square\end{aligned}$$

3.3. Example. A perfect map which does not preserve Δ-normality.

We shall construct an upper semicontinuous decomposition of the space Y from Example 3.2 into one- and two-point subsets such that the quotient space Z is not Δ-normal. First of all note that the x-axis A is a closed and discrete subset of Y, so that any decomposition of A induces a u.s.c. decomposition of Y.

We define a decomposition of $\boldsymbol{R}$, and hence of A, as follows.

Let $\{K_\alpha | \alpha \in 2^\omega\}$ be an enumeration of the collection of all countable unions of closed nowhere dense subsets of $\boldsymbol{R}^2$ which contain $\Delta\boldsymbol{R}$. Inductively choose for all $\alpha \in 2^\omega$ a point

$$\langle x_\alpha, y_\alpha\rangle \in \boldsymbol{P}^2 \setminus \Big(K_\alpha \cup \bigcup_{\beta \in \alpha} (\{x_\beta, y_\beta\} \times \boldsymbol{R} \cup \boldsymbol{R} \times \{x_\beta, y_\beta\})\Big).$$

To see that this is possible consider for some $\alpha \in 2^\omega$

$$A_\alpha = \{x \in \boldsymbol{R} |\ K_\alpha[x] \text{ is of second category in } \boldsymbol{R}\}.$$

We claim that A_α is of first category in $\boldsymbol{R}$.

Suppose not. Let $K_\alpha = \bigcup_{i \in \omega} K_i$ with each K_i closed and nowhere dense and let $\{B_n\}_{n \in \omega}$ be a base for $\boldsymbol{R}$.

Let $A_{i,n} = \{x \in \boldsymbol{R} |\ B_n \subseteq \overline{K_i[x]}\}$, $i, n \in \omega$. Then $A_\alpha = \bigcup_{i,n} A_{i,n}$ by definition of A_α. So by assumption $\overline{A_{i,n}}^\circ \neq \emptyset$ for some i and n, hence $B_m \subseteq \bar{A}_{i,n}$ for some m. But then $B_m \times B_n \subseteq K_i$ contradicting the fact that K_i is nowhere dense.

So since A_α is of first category, $|\boldsymbol{P} \setminus A_\alpha| = 2^\omega$, hence we can pick

$$x_\alpha \in \boldsymbol{P} \setminus \Big(A_\alpha \cup \bigcup_{\beta \in \alpha} \{x_\beta, y_\beta\}\Big)$$

and since $K_\alpha[x_\alpha]$ is of first category we can pick

$$y_\alpha \in \boldsymbol{P} \setminus \Big(K_\alpha[x_\alpha] \cup \bigcup_{\beta \in \alpha} \{x_\beta, y_\beta\} \cup \{x_\alpha\}\Big).$$

Clearly $\langle x_\alpha, y_\alpha\rangle$ is as required. The set $E = \{\langle x_\alpha, y_\alpha\rangle | \alpha \in 2^\omega\}$ is of second category in $\boldsymbol{R}^2$. Let Z be the quotient space obtained from Y by identifying the points $\langle x_\alpha, 0\rangle$ and $\langle y_\alpha, 0\rangle$ for all $\alpha \in 2^\omega$. As noted above the projection $\pi\colon Y \to Z$ is closed and since the fibers have at most two points π is perfect. We now show that Z is not Δ-normal. To this end we put $V_n(x) = \pi[U_n(x)]$ if $x \notin \bigcup_{\alpha \in 2^\omega} \{x_\alpha, y_\alpha\}$ and $V_n(z_\alpha) = \pi[U_n(x_\alpha) \cup U_n(y_\alpha)]$ where $z_\alpha = \pi(\langle x_\alpha, 0\rangle)$ $= \pi(\langle y_\alpha, 0\rangle)$ for $\alpha \in 2^\omega$. Then $\{V_n(z)\}_{n \in N}$ is a local base at z for all $z \in \pi[A]$. Let $\mathscr{U} = \{\pi[Y \setminus A]\} \cup \{V_1(z)\}_{z \in A}$. $\mathscr{U}$ is an open cover of Z. Let $\mathscr{V}$ be an open refinement of $\mathscr{U}$. For all $z \in \pi[A]$ pick $n_z \in N$ such that $V_{n_z}(z) \subseteq$ some $V \in \mathscr{V}$.

Let

$$E_n = \{\langle x, y\rangle \in E |\ n_{\pi(\langle x, 0\rangle)} = n_{\pi(\langle y, 0\rangle)} = n\}, \qquad n \in N.$$

For some $n \in N$ we have that $\bar{E}_n^\circ \neq \emptyset$, since E is of second category and $E = \bigcup_{n=1}^{\infty} E_n$. Take $\langle p, q\rangle \in Q^2$ such that $\langle p-1, q-1\rangle \in \bar{E}_n^\circ$ and $\varepsilon > 0$ such that $(p-1-\varepsilon, p-1+\varepsilon) \times (q-1-\varepsilon, q-1+\varepsilon) \subseteq \bar{E}_n$. Let $m \in N$ be arbitrary and choose $\langle x_\alpha, y_\alpha\rangle \in E_n$ such that $p-1-\delta < x_\alpha < p-1$ and $q-1-\delta < y_\alpha < p-1$ where $\delta = \min\{\varepsilon, 1/n, 1/m\}$. It is easy to see that $U_n(x_\alpha) \cap U_m(p) \neq \emptyset$ and $U_n(y_\alpha) \cap U_m(q) \neq \emptyset$ and $V_n(z_\alpha) \cap V_m(q) \neq \emptyset$, hence we can find a $V \in \mathscr{V}$ such that $V_m(p) \cap V \neq \emptyset \neq V_m(q) \cap V$. Since m was arbitrary and since clearly $\forall U \in \mathscr{U}$: $p \notin U \vee q \notin U$ we see that $\mathscr{V}$ does not have the property described in Theorem 2.1. We conclude that Z is not Δ-normal. □

3.4. Example. A Δ-normal space X and a compact space Y such that $X \times Y$ is not Δ-normal.

Let $X = \omega_1$ and $Y = \omega_1 + 1$. X is a linearly ordered topological space (LOTS), hence divisible [9], hence Δ-normal. Y is a compact LOTS and hence Δ-normal for two reasons. Let $A = \{\langle\alpha, \omega_1\rangle \mid \alpha \in \omega_1\}$ and $B = \{\langle\alpha, \alpha\rangle \mid \alpha \in \omega_1\}$. A and B are closed and disjoint subsets of $X \times Y$, which, as is well known, cannot be separated by open sets. However A and B do satisfy the requirements of Lemma 2.3, so $X \times Y$ cannot be Δ-normal. □

Example 3.4 shows that the perfect preimage of a Δ-normal space need not be Δ-normal. We can do a little better, in fact we can give an example of an at most two-to-one closed irreducible map of a non Δ-normal space onto a Δ-normal space.

3.5. Example. Consider the subspace $C = \{\langle\alpha, \beta\rangle \mid \beta \geqslant \alpha\}$ of $\omega_1 \times (\omega_1 + 1)$. In 3.4 we actually showed that C is not Δ-normal. We now employ a trick used by J. Vermeer [13] to give an easy example of a normal space with a nonnormal absolute.

For all limit ordinals α in ω_1 we identify $\langle\alpha, \alpha\rangle$ and $\langle\alpha, \omega_1\rangle$. Call the quotient space Z. As in [13] the projection map π: $C \to Z$ is closed and irreducible, it clearly has fibers with at most two points only. Let $\mathscr{U}$ be an open cover of Z. For all limits α in ω_1 pick $\gamma_\alpha < \alpha$ and $\beta_\alpha \in \omega$ such that

$$(\gamma_\alpha, \alpha] \times ((\gamma_\alpha, \alpha] \cup (\beta_\alpha, \omega_1]) \subseteq \pi^{\leftarrow}[U] \quad \text{for some } U \in \mathscr{U}.$$

Take an unbounded set S in ω_1 and an ordinal γ such that for all $\alpha \in S$ $\gamma_\alpha = \gamma$. Let $A = \{\langle\alpha, \beta\rangle \mid \alpha \leqslant \gamma\}$ and $B = \{\langle\alpha, \beta\rangle \mid \alpha > \gamma\}$. A and B are disjoint, clopen and they cover C. So the same holds for $\pi[A]$ and $\pi[B]$ with respect to Z. $\pi[A]$ is compact and hence Δ-normal so let $\mathscr{V}_0$ be an open refinement of $\{U \cap \pi[A] \mid U \in \mathscr{U}\}$ as in Theorem 2.1 and let $\mathscr{V}_1 = \{U \cap \pi[B] \mid U \in \mathscr{U}\}$. $\mathscr{V} = \mathscr{V}_0 \cup \mathscr{V}_1$ is a refinement of $\mathscr{U}$. A moment's reflection will show that if $x, y \in Z$ satisfy $\forall U \in \mathscr{U}$: $x \notin U \vee y \notin U$ then $x, y \in A$ or ($x \in A$ and $y \in B$). In the first case take $O_x \ni x$ and $O_y \ni y$ such that $\forall V \in \mathscr{V}_0$: $O_x \cap V = \emptyset$ or $O_y \cap V = \emptyset$ and $O_x, O_y \subseteq A$, in the second case let $O_x = A$ and $O_y = B$. In both cases $\forall V \in \mathscr{V}$: $O_x \cap V = \emptyset \vee O_y \cap V = \emptyset$. □

The above examples suggest some questions. First of all, in Example 3.3, the domain of the map is not very nice, so the question arises whether the perfect image of a nice (i.e. normal, countably paracompact, etc.) Δ-normal space is again Δ-normal (**P 1318**).

Furthermore, Example 3.4 suggests the question of whether the product of a Δ-normal space with for example the unit interval or a convergent sequence is Δ-normal (**P 1319**). Finally, since Δ-normality is independent of normality etc., it would be interesting to know what extra properties, for example a normal space has, if it is also Δ-normal. In this context let us mention that a Δ-normal Dowker space exists: in [5] the author showed that M. E. Rudin's Dowker Space [11] is, in fact, divisible.

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