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A CHARACTERIZATION OF A CLASS OF CONVOLUTIONS

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1. Consider the class $\mathfrak{P}$ of all probability measures defined on Borel subsets of the positive half-line. By E_u $(u \geqslant 0)$ we shall denote the probability measure concentrated at the point u. For any positive number c we define a transformation T_c of $\mathfrak{P}$ onto itself by means of the formula $(T_c P)(\mathscr{A}) = P(c^{-1}\mathscr{A})$, where $P \in \mathfrak{P}$, $\mathscr{A}$ is a Borel set and $c^{-1}\mathscr{A} = \{c^{-1}x: x \in \mathscr{A}\}$. Further, we define the transformation T_0 by assuming $T_0 P = E_0$ for all $P \in \mathfrak{P}$. We say that a sequence $P_1, P_2, \ldots$ of probability measures is *weakly convergent to a probability measure* P, in symbols $P_n \to P$, if for every bounded continuous function φ the equation

$$\lim_{n\to\infty} \int_0^\infty \varphi(x) P_n(dx) = \int_0^\infty \varphi(x) P(dx)$$

holds.

A commutative and associative $\mathfrak{P}$-valued binary operation $\circ$ defined on $\mathfrak{P}$ is called a *generalized convolution* if it satisfies the following conditions:

(i) the measure E_0 is a unit element, i.e. $E_0 \circ P = P$ for all $P \in \mathfrak{P}$;

(ii) $(aP + bQ)\circ R = a(P\circ R) + b(Q\circ R)$, whenever $P, Q, R \in \mathfrak{P}$ and $a \geqslant 0$, $b \geqslant 0$, $a+b = 1$;

(iii) $(T_c P)\circ(T_c Q) = T_c(P\circ Q)$ for any $P, Q \in \mathfrak{P}$ and $c \geqslant 0$;

(iv) if $P_n \to P$, then $P_n \circ Q \to P\circ Q$ for all $Q \in \mathfrak{P}$;

(v) there exists a sequence $c_1, c_2, \ldots$ of positive numbers such that the sequence $T_{c_n} E_1^{\circ n}$ weakly converges to a measure different from E_0 (the law of large numbers for measures concentrated at a single point).

The power $E_c^{\circ n}$ is taken in the sense of the operation $\circ$, i.e. $E_c^{\circ 1} = E_c$, $E_c^{\circ n} = E_c^{\circ(n-1)} \circ E_c$ $(n = 2, 3, \ldots)$. For a general theory of generalized convolutions as well as examples of generalized convolutions we refer to the paper [2]. We shall quote only two examples of generalized convolutions. In both examples generalized convolutions $P\circ Q$ will be defined by means of the functional $\int_0^\infty \varphi(x)(P\circ Q)(dx)$ on all bounded continuous functions φ.

a-convolution $(0 < a < \infty)$:

$$\int_0^\infty \varphi(x)(P\circ Q)(dx) = \int_0^\infty\int_0^\infty \varphi\big((x^a+y^a)^{1/a}\big)P(dx)Q(dy). \tag{1.1}$$

For $a = 1$ we obtain the ordinary convolution.

$(a, 1)$-*convolution* $(0 < a < \infty)$:

$$\int_0^\infty \varphi(x)(P\circ Q)(dx) = \frac{1}{2}\int_0^\infty\int_0^\infty \big[\varphi\big((x^a+y^a)^{1/a}\big)+\varphi(|x^a-y^a|^{1/a})\big]P(dx)Q(dy). \tag{1.2}$$

The present paper is devoted to a simple characterization of a-convolutions and $(a, 1)$-convolutions.

The class $\mathfrak{P}$ with a generalized convolution $\circ$ will be called a *generalized convolution algebra* and denoted by $(\mathfrak{P}, \circ)$. A continuous mapping h of $\mathfrak{P}$ into the real field is called a *homomorphism of the algebra* $(\mathfrak{P}, \circ)$ if $h(aP+bQ) = ah(P)+bh(Q)$, whenever $a \geqslant 0$, $b \geqslant 0$, $a+b = 1$, and $h(P\circ Q) = h(P)h(Q)$ for all $P, Q \in \mathfrak{P}$. Of course, each generalized convolution algebra admits two trivial homomorphisms $h(P) \equiv 0$ and $h(P) \equiv 1$. Algebras admitting a non-trivial homomorphism will be called *regular*. In this case the generalized convolution will be also called *regular*. Both a-convolution and $(a, 1)$-convolution are regular (see [2], Section 2).

We say that an algebra $(\mathfrak{P}, \circ)$ admits a *characteristic function* if there exists a one-to-one correspondence $P \leftrightarrow \Phi_P$ between probability measures P from $\mathfrak{P}$ and real-valued functions Φ_P defined on the positive half-line such that $\Phi_{aP+bQ} = a\Phi_P + b\Phi_Q$ $(a \geqslant 0,\ b \geqslant 0,\ a+b = 1)$, $\Phi_{P\circ Q} = \Phi_P\Phi_Q$, $\Phi_{T_aP}(t) = \Phi_P(at)$ $(a \geqslant 0,\ t \geqslant 0)$ and the uniform convergence in every finite interval of Φ_{P_n} is equivalent to the weak convergence of P_n. The function Φ_P will be called the *characteristic function* of the probability measure P in the generalized convolution algebra $(\mathfrak{P}, \circ)$. The characteristic function in generalized convolution algebras plays the same fundamental role as in ordinary convolution algebra, i.e. in classical problems concerning the addition of independent random variables.

It was proved in [2] (Theorem 6) that a generalized convolution algebra admits a characteristic function if and only if it is regular. Moreover, each characteristic function Φ_P is an integral transform,

$$\Phi_P(t) = \int_0^\infty \Omega(tx)P(dx), \tag{1.3}$$

where the kernel Ω satisfies the condition

$$\lim_{x\to 0}\frac{1-\Omega(tx)}{1-\Omega(x)} = t^\varkappa \qquad (t \geqslant 0) \tag{1.4}$$

for some positive number $\varkappa$ called the *characteristic exponent* of the generalized convolution algebra. For instance, $\Omega(t) = \exp(-t^{a})$ and $\Omega(t) = \cos(t^{a})$ are kernels of characteristic functions in a-convolution and $(a, 1)$-convolution algebras respectively.

2. Let $s(P)$ denote the support of the measure P and $|\mathscr{A}|$ the cardinal number of the set $\mathscr{A}$. Since $s(E_x \circ E_y) = \{(x^a + y^a)^{1/a}\}$ for a-convolutions and $s(E_x \circ E_y) = \{(x^a + y^a)^{1/a}, |x^a - y^a|^{1/a}\}$ for $(a, 1)$-convolutions, we infer that a-convolutions and $(a, 1)$-convolutions fulfill the following conditions:

$$(*) \qquad |s(E_x \circ E_y)| \leqslant 2 \quad \text{for all } x, y \geqslant 0,$$

$$(**) \qquad \text{the set } \bigcup_{0 \leqslant x, y \leqslant 1} s(E_x \circ E_y) \text{ is bounded.}$$

These conditions characterize a-convolutions and $(a, 1)$-convolutions among regular generalized convolutions. Namely, we shall prove the following theorem:

THEOREM 1. *A regular generalized convolution satisfies conditions* $(*)$ *and* $(**)$ *if and only if it is either an a-convolution or an $(a, 1)$-convolution.*

Theorem 1 is a direct consequence of the following two theorems:

THEOREM 2. *If a regular generalized convolution satisfies the condition $|s(E_u \circ E_v)| = 1$ for some positive numbers u and v, then it is an a-convolution.*

THEOREM 3. *If a regular generalized convolution satisfies condition* $(**)$ *and the equation $|s(E_x \circ E_y)| = 2$ for all positive numbers x and y, then it is an $(a, 1)$-convolution.*

3. In this Section we shall prove Theorem 2. First of all we shall prove two Lemmas.

LEMMA 3.1. *If $\circ$ is a regular generalized convolution and $P \circ Q = E_0$, then $P = Q = E_0$.*

Proof. Let Φ_P be a characteristic function in the generalized convolution algebra in question. Suppose that $P \circ Q = E_0$. Then we have the formula $\Phi_P \Phi_Q = 1$. Since $|\Phi_P(t)| \leqslant 1$ and $|\Phi_Q(t)| \leqslant 1$ for all $t \geqslant 0$ (see [2], Theorems 1 and 6), we have the equation $|\Phi_P(t)| = |\Phi_Q(t)| = 1$ for all $t \geqslant 0$. Hence, taking into account the continuity of characteristic functions and the formula $\Phi_P(0) = \Phi_Q(0) = 1$, we get the equation $\Phi_P = \Phi_Q = 1$. Thus $P = Q = E_0$ which completes the proof.

LEMMA 3.2. *If $\circ$ is a regular generalized convolution and Q a probability measure with positive characteristic function, then the equation $P \circ Q = E_a$ implies the equation $P = E_b$, where b is a non-negative number.*

Proof. Suppose that the measure P is not concentrated at a single point. Then there are two different measures P_1 and P_2 in P such that $P = \frac{1}{2}(P_1+P_2)$. Hence we get the formula $\frac{1}{2}P_1 \circ Q + \frac{1}{2}P_2 \circ Q = E_a$. Thus both measures $P_1 \circ Q$ and $P_2 \circ Q$ are concentrated at the point a and, consequently, $P_1 \circ Q = P_2 \circ Q = E_a$. The last equation implies the equation $\Phi_{P_1}\Phi_Q = \Phi_{P_2}\Phi_Q$ for characteristic functions. Since, by the assumption, the function Φ_Q is positive, we obtain the formula $\Phi_{P_1} = \Phi_{P_2}$ which implies $P_1 = P_2$. But this contradicts the inequality $P_1 \neq P_2$. The Lemma is thus proved.

Proof of the Theorem 2. The assumption $|s(E_u \circ E_v)| = 1$ can be written in the form $E_u \circ E_v = E_w$, where, by Lemma 3.1, $w > 0$. Moreover, by (iii), we may assume that $w = 1$, i.e. $E_u \circ E_v = E_1$. Since, by (1.3), $\Phi_{E_x}(t) = \Omega(xt)$, we get

$$\Omega(t) = \Omega(ut)\Omega(vt) \qquad (t \geqslant 0). \tag{3.1}$$

Hence, by (1.4), we get the formula

$$t^{\varkappa} = \lim_{x\to 0} \frac{1-\Omega(tx)}{1-\Omega(x)} = \lim_{x\to 0} \frac{1-\Omega(utx)\Omega(vtx)}{1-\Omega(x)} = (ut)^{\varkappa} + (vt)^{\varkappa}.$$

Thus $u < 1$ and $v < 1$.

Now we shall prove that the function Ω is positive. Since $\Omega(0) = 1$, it suffices to prove that Ω is different from 0 everywhere. Suppose the contrary. Let t_0 be the smallest positive number satisfying the equation $\Omega(t_0) = 0$. Taking into account (3.1), we infer that $\Omega(ut_0) = 0$ or $\Omega(vt_0) = 0$ which contradicts the inequalities $u < 1$ and $v < 1$. Thus

$$\Omega(t) > 0 \qquad (t \geqslant 0). \tag{3.2}$$

Hence we get the inequality $\Phi_{E_{v^2}}(t) = \Omega(v^2 t) > 0$. Further, from the formula

$$E_1 = E_u \circ E_v = E_u \circ T_v E_1 = E_u \circ (T_v E_u \circ T_v E_v) = (E_u \circ E_{uv}) \circ E_{v^2}$$

and Lemma 3.2 it follows that $E_u \circ E_{uv} = E_z$, where, by Lemma 3.1, z is a positive number. Consequently,

$$E_1 \circ E_{v^{-1}} = T_{(uv)^{-1}}(E_u \circ E_{uv}) = E_p, \tag{3.3}$$

where $p = z(uv)^{-1}$. Taking into account the formula

$$E_{v^{-1}} = T_{v^{-1}} E_1 = T_{v^{-1}}(E_u \circ E_v) = E_a \circ E_1,$$

where $a = uv^{-1}$, we get from (3.3) the equation $E_p = E_1^{\circ 2} \circ E_a$. Since, by (3.2), $\Phi_{E_a}(t) = \Omega(at) > 0$ $(t \geqslant 0)$, the last equation and Lemma 3.2 imply the formula $E_1^{\circ 2} = E_q$, where, by Lemma 3.1, q is a positive number.

Now we shall prove by induction the formula

$$E_1^{\circ n} = E_{a_n}, \tag{3.4}$$

where a_n is a positive number. Suppose that $k \geqslant 2$ and $E_1^{\circ k} = E_{a_k}$, where $a_k > 0$. Setting $Q = E_1^{\circ(k-1)}$ we have

$$E_1^{\circ(k+1)} \circ Q = E_1^{\circ 2k} = E_{a_k}^{\circ 2} = T_{a_k} E_1^{\circ 2} = T_{a_k} E_q = E_{qa_k}.$$

Moreover, by (3.2), $\Phi_Q(t) = \Omega(t)^{k-1} > 0$ $(t \geqslant 0)$. Hence, by Lemmas 3.1 and 3.2, we get the formula $E_1^{\circ(k+1)} = E_{a_{k+1}}$, where a_{k+1} is a positive number. Formula (3.4) is thus proved.

Setting $c_n = a_n^{-1}$ we get, in view of (3.4), the formula $E_1 = T_{c_n} E_{a_n} = T_{c_n} E_1^{\circ n}$ $(n = 1, 2, \ldots)$. Consequently, by Theorem 15 in [2], E_1 is a stable measure. Since the measure E_1 is purely atomic, the generalized convolution $\circ$ is, by Theorem 17 in [2], an α-convolution. Theorem 2 is thus proved.

4. Before proving Theorem 3 we shall prove some lemmas. From now on we will make the assumption that the generalized convolution $\circ$ is regular, satisfies condition (**) and the equation

$$|s(E_x \circ E_y)| = 2 \qquad (x, y > 0). \tag{4.1}$$

For such generalized convolutions we have the formula

$$E_x \circ E_y = a(x, y) E_{f(x,y)} + \big(1 - a(x, y)\big) E_{g(x,y)} \qquad (x, y > 0), \tag{4.2}$$

$$0 < a(x, y) < 1 \tag{4.3}$$

and

$$f(x, y) > g(x, y). \tag{4.4}$$

Further, by condition (iv), the functions $a(x, y)$, $f(x, y)$ and $g(x, y)$ are continuous with respect to each variable $x, y > 0$. Since the generalized convolution is commutative, all these functions are symmetrical. Moreover, by condition (iii), we have

$$a(zx, zy) = a(x, y), \tag{4.5}$$

$$f(zx, zy) = zf(x, y) \tag{4.6}$$

and

$$g(zx, zy) = zg(x, y) \tag{4.7}$$

for all positive numbers x, y and z. Finally, by (**), for all $x > 0$, the inequality

$$\varlimsup_{y \to 0} f(x, y) < \infty \tag{4.8}$$

holds.

From (ii), (4.2) and (4.3) it follows that

$$(4.9)\qquad s\big((E_x\circ E_y)\circ E_z\big) = \big\{f\big(f(x,y),z\big),\ f\big(g(x,y),z\big),\ g\big(f(x,y),z\big),\ g\big(g(x,y),z\big)\big\}$$

for all positive numbers x, y and z. Hence and from (4.4) we obtain the formula $\sup s\big((E_x\circ E_y)\circ E_z\big) = \max\Big(f\big(f(x,y),z\big),\ f\big(g(x,y),z\big)\Big)$. This formula and the commutative and associative laws for generalized convolutions yields the functional equation

$$(4.10)\qquad \max\Big(f\big(f(x,y),z\big),\ f\big(g(x,y),z\big)\Big) = \max\Big(f\big(f(y,z),x\big),\ f\big(g(y,z),x\big)\Big).$$

LEMMA 4.1. *The functions $a(x,y)$, $f(x,y)$ and $g(x,y)$ satisfy for $x, y > 0$ the equation*

$$a(x,y)f(x,y)^{\varkappa}+\big(1-a(x,y)\big)g(x,y)^{\varkappa} = x^{\varkappa}+y^{\varkappa},$$

where $\varkappa$ is the characteristic exponent of the generalized convolution algebra.

Proof. By the definition of the characteristic function and its integral representation (1.3) we have

$$\Phi_{E_x\circ E_y}(t) = \Phi_{E_x}(t)\Phi_{E_y}(t) = \Omega(xt)\Omega(yt).$$

On the other hand, by (4.2),

$$\Phi_{E_x\circ E_y}(t) = a(x,y)\Omega\big(f(x,y)t\big)+\big(1-a(x,y)\big)\Omega\big(g(x,y)t\big).$$

The first equation and (1.4) imply

$$\lim_{t\to 0}\frac{1-\Phi_{E_x\circ E_y}(t)}{1-\Omega(t)} = x^{\varkappa}+y^{\varkappa}.$$

Further, the second equation and (1.4) yield the formula

$$\lim_{t\to 0}\frac{1-\Phi_{E_x\circ E_y}(t)}{1-\Omega(t)} = a(x,y)f(x,y)^{\varkappa}+\big(1-a(x,y)\big)g(x,y)^{\varkappa}$$

whence the assertion of the Lemma follows.

From Lemma 4.1 and formulas (4.3) and (4.4) we get the following corollary:

$$(4.11)\qquad f(x,y) > (x^{\varkappa}+y^{\varkappa})^{1/\varkappa}\qquad (x, y > 0).$$

LEMMA 4.2. *For all $x > 0$ the formula*

$$\lim_{y\to 0} f(x,y) = x$$

holds.

Proof. Setting $m = \overline{\lim\limits_{y\to 0}} f(1, y)$, we have, by (4.8) and (4.11), the inequality $1 \leqslant m < \infty$. Moreover, by (4.6),

$$\overline{\lim_{y\to 0}} f(x, y) = mx \qquad (x > 0). \tag{4.12}$$

Consequently, we can choose a sequence $\{z_k\}$ of positive numbers tending to 0 such that

$$\lim_{k\to\infty} f(m, z_k) = m^2. \tag{4.13}$$

Let $\{y_n\}$ be a sequence of positive numbers tending to 0 such that

$$\lim_{n\to\infty} f(1, y_n) = m. \tag{4.14}$$

We can also assume that both limits

$$\lim_{n\to\infty} f(y_n, z_k) = u_k \qquad (k = 1, 2, \ldots), \tag{4.15}$$

$$\lim_{n\to\infty} g(y_n, z_k) = v_k \qquad (k = 1, 2, \ldots) \tag{4.16}$$

exist. By (4.4) and (4.12) we have the inequality $0 \leqslant v_k \leqslant u_k \leqslant mz_k$ $(k = 1, 2, \ldots)$. Consequently, both sequences $\{u_k\}$ and $\{v_k\}$ tend to 0. Hence and from (4.12) it follows that

$$\overline{\lim_{k\to\infty}} \lim_{n\to\infty} f\big(f(y_n, z_k), 1\big) \leqslant m \tag{4.17}$$

and

$$\overline{\lim_{k\to\infty}} \lim_{n\to\infty} f\big(g(y_n, z_k), 1\big) \leqslant m. \tag{4.18}$$

Further, by (4.10), we have the inequality

$$f\big(f(1, y_n), z_k\big) \leqslant \max\Big(f\big(f(y_n, z_k), 1\big), f\big(g(y_n, z_k), 1\big)\Big) \qquad (n, k = 1, 2, \ldots).$$

Hence and from (4.13), (4.14), (4.17) and (4.18) we obtain the inequality

$$m^2 = \lim_{k\to\infty} \lim_{n\to\infty} f\big(f(1, y_n), z_k\big) \leqslant m$$

which together with the inequality $m \geqslant 1$ yields $m = 1$. Thus, by (4.12), $\overline{\lim\limits_{y\to 0}} f(x, y) = x$ for all $x > 0$. The inequality $\underline{\lim\limits_{y\to 0}} f(x, y) \geqslant x$ is a direct consequence of inequality (4.11), which completes the proof.

Lemma 4.3. *There exists a positive number a such that $f(x, y) = (x^a + y^a)^{1/a}$. Moreover, the number a satisfies the inequality $a < \varkappa$ where $\varkappa$ is the characteristic exponent of the generalized convolution algebra.*

Proof. First we shall prove the inequality

$$f(x, y_1) > f(x, y_2) \quad \text{for} \quad x > 0 \text{ and } y_1 > y_2 > 0. \tag{4.19}$$

Suppose the contrary. Then there are positive numbers $u_1 > u_2$ for which the inequality $f(x, u_1) \leqslant f(x, u_2)$ holds. By (4.11), for all $y > 0$ we have $f(x, y) > x$. Further, the function f is continuous with respect to each variable and, by Lemma 4.2, $\lim_{y\to 0} f(x, y) = x$. Consequently, we can choose two numbers v_1 and v_2 satisfying the inequality $v_1 > v_2 > 0$ for which

$$f(x, v_1) = f(x, v_2) = \max_{0<y\leqslant v_1} f(x, y). \tag{4.20}$$

Taking into account the symmetry of the function f we have, in view of (4.11) and Lemma 4.2, the relations

$$\lim_{u\to 0} f(u, v_2) = v_2, \qquad \lim_{u\to\infty} f(u, v_2) = \infty.$$

Hence, by the continuity of f with respect to the first variable, we get the existence of a positive number z satisfying the equation

$$f(z, v_2) = v_1. \tag{4.21}$$

By (4.4) the number v_1 satisfies the inequality $v_1 > g(z, v_2)$. Thus, by (4.20), $f(x, v_1) \geqslant f\big(x, g(z, v_2)\big)$. Moreover, according to (4.21), $f(x, v_1) = f\big(x, f(z, v_2)\big)$ and, consequently,

$$f(x, v_1) = \max\Big(f\big(x, f(z, v_2)\big),\ f\big(x, g(z, v_2)\big)\Big).$$

Hence, taking into account the symmetry of the function f and equation (4.10), we get the equation

$$f(x, v_1) = \max\Big(f\big(f(x, v_2), z\big),\ f\big(g(x, v_2), z\big)\Big).$$

Thus, by (4.11) and (4.20), we have the inequality

$$f(x, v_1) \geqslant f\big(f(x, v_2), z\big) = f\big(f(x, v_1), z\big) > \big(f(x, v_1)^\varkappa + z^\varkappa\big)^{1/\varkappa} > f(x, v_1)$$

which gives the contradiction. Inequality (4.19) is thus proved.

From (4.4), (4.10) and (4.19) it follows that the function f satisfies the equation

$$f\big(f(x, y), z\big) = f\big(f(y, z), x\big) \quad (x, y, z > 0). \tag{4.22}$$

Moreover, by the symmetry of f and formula (4.6), we have the equations

$$f(x, y) = f(y, x), \quad f(zx, zy) = zf(x, y) \quad (x, y, z > 0). \tag{4.23}$$

Further, by Lemma 4.2, the function f satisfies the boundary condition

(4.24) $$\lim_{y\to 0} f(x, y) = x.$$

F. Bohnenblust proved in [1] (p. 630-632) that each function f satisfying (4.19), (4.22), (4.23) and (4.24) is of the form $f(x, y) = (x^a + y^a)^{1/a}$, where a is a positive number. Moreover, from (4.11) the inequality $a < \varkappa$ follows. The Lemma is thus proved.

LEMMA 4.4 *The function $a(x, y)$ is constant for $x, y > 0$.*

Proof. From formulae (4.4), (4.9) and Lemma 4.3 it follows that $\sup s\big((E_x \circ E_y) \circ E_z\big) = \{f(f(x, y), z)\}$ for all $x, y, z > 0$. Consequently, by (4.2), the measure $(E_x \circ E_y) \circ E_z$ can be written in the form

$$(E_x \circ E_y) \circ E_z = a(x, y)\, a\big(f(x, y), z\big) E_{f(f(x,y),z)} + \big(1 - a(x, y)\, a(f(x, y), z)\big) R,$$

where R is a probability measure satisfying the condition $\sup s(R) < f\big(f(x, y), z\big)$. Hence, by the commutative and associative laws for generalized convolutions, we get the equation

(4.25) $$a(x, y)\, a\big(f(x, y), z\big) = a(y, z)\, a\big(f(y, z), x\big) \quad (x, y, z > 0).$$

Put $F(x) = a(x^{1/a}, 1)$, where the constant a is determined by Lemma 4.3. By (4.5) to prove the Lemma it suffices to prove that the function $F(x)$ is constant for $x > 0$.

The function $F(x)$ is continuous and, by (4.3), positive for $x > 0$. Moreover, by the symmetry of the function $a(x, y)$ and (4.5), we have the equation

(4.26) $$F(x) = F(x^{-1}) \quad (x > 0).$$

Setting $x = u^{1/2}$, $y = 1$ and $z = (1 + u)^{1/2}$ $(u > 0)$ into (4.25) and taking into account (4.5) and Lemma 4.3, we get the equation

$$F(u)\, F\big((1 + u)^{1/2}\big) = F\big((1 + u)^{1/2}\big)\, F\big(u^{-1}(1 + (1 + u)^{1/2})\big).$$

Since the function F is positive everywhere, the last equation implies the equation

(4.27) $$F(u) = F\big(G(u)\big) \quad (u > 0),$$

where $G(u) = u^{-1}\big(1 + (1 + u)^{1/2}\big)$. Put $H(u) = G\big(G(u^{-1})\big)^{-1}$. From (4.26) and (4.27) we obtain the equation

(4.28) $$F(x) = F\big(H(x)\big) \quad (x > 0).$$

Let v be an arbitrary number satisfying the inequality $v \geqslant (\sqrt{5} - 1)/2$. Put $x_1 = v$ and $x_{n+1} = H(x_n)$ $(n = 1, 2, \ldots)$. Since the function H is monotone increasing and $H\big((\sqrt{5} - 1)/2\big) = (\sqrt{5} - 1)/2$, we have the inequality $x_n \geqslant (\sqrt{5}-1)/2$ $(n = 1, 2, \ldots)$. Moreover, $H(x) \leqslant x$ for $x \geqslant (\sqrt{5}-1)/2$. Hence we get the inequality $x_{n+1} \leqslant x_n$ $(n = 1, 2, \ldots)$. Thus the sequence $\{x_n\}$ is convergent to a limit $q \geqslant (\sqrt{5}-1)/2$. Of course, this limit satisfies the equation $q = H(q)$. Since $(\sqrt{5}-1)/2$ is the only positive solution of the last equation, we have $\lim_{n\to\infty} x_n = (\sqrt{5}-1)/2$. Hence and from (4.28), by virtue of the continuity of the function F, we get $F(v) = F\big((\sqrt{5}-1)/2\big)$ for all numbers v satisfying the condition $v \geqslant (\sqrt{5}-1)/2$. Now equation (4.26) implies that F is a constant function. The Lemma is thus proved.

Proof of the Theorem 3. By Lemma 4.4 there exists a constant c such that $a(x, y) = c$ for all $x, y > 0$. Moreover, by (4.3), $0 < c < 1$. Further, by Lemmas 4.1 and 4.3,

$$g(x, y)^{\varkappa} = (1-c)^{-1}[x^{\varkappa}+y^{\varkappa}-c(x^{a}+y^{a})^{\varkappa/a}], \tag{4.29}$$

where $\varkappa$ is the characteristic exponent of the generalized convolution algebra in question and $0 < a < \varkappa$. From (4.4), (4.9) and Lemma 4.3 we get the equation

$$\begin{aligned} \max\big(f(g(x, y), z),\ g(f(x, y), z)\big) &= \sup\Big(s\big((E_x \circ E_y) \circ E_z\big) \setminus \big\{f\big(f(x, y), z\big)\big\}\Big) \\ &= \sup\Big(s\big((E_y \circ E_z) \circ E_x\big) \setminus \big\{f\big(f(y, z), x\big)\big\}\Big) \\ &= \max\big(f(g(y, z), x),\ g(f(y, z), x)\big). \end{aligned} \tag{4.30}$$

Further, by (4.29) and Lemma 4.3, we have the inequalities $f\big(g(x, y), z\big) > g\big(f(x, y), z\big)$ and $f\big(g(y, z), x\big) > g\big(f(y, z), x\big)$ for sufficiently small positive numbers y. Thus, by (4.30), $f\big(g(x, y), z\big) = f\big(g(y, z), x\big)$ for sufficiently small y. Consequently, setting

$$\varrho = \varkappa/a, \qquad U(t) = (1-c)^{1/\varrho} f\big(g(1, t^{1/a}), 2^{1/a}\big)^{a},$$
$$V(t) = (1-c)^{1/\varrho} f\big(g(t^{1/a}, 2^{1/a}), 1\big)^{a},$$

we have

$$U(t) = V(t) \qquad (0 < t < t_0), \tag{4.31}$$

where t_0 is a sufficiently small positive number. It is clear that $\varrho > 1$ and the functions $U(t)$ and $V(t)$ are twice differentiable for $t > 0$. If $\varrho \neq 2$, then

$$\lim_{t\to 0} \frac{d^2 V}{dt^2} \Big/ \frac{d^2 U}{dt^2} = 2^{1-\varrho}$$

which together with the inequality $\varrho > 1$ contradicts (4.31). Thus $\varrho = 2$ and, consequently, by (4.31),

$$(1-2c)(1-c)^{-3/2} = \lim_{t\to 0} \frac{d^2 U}{dt^2} = \lim_{t\to 0} \frac{d^2 V}{dt^2} = 2^{-1}(1-2c)(1-c)^{-3/2}.$$

Hence $c = 1/2$. Now formula (4.29) yields $g(x, y) = |x^a - y^a|^{1/a}$. Consequently, by (4.2),

$$E_x \circ E_y = \tfrac{1}{2} E_{(x^a+y^a)^{1/a}} + \tfrac{1}{2} E_{|x^a-y^a|^{1/a}} \qquad (x, y > 0).$$

Thus formula (1.2) holds if $P = E_x$ and $Q = E_y$ $(x, y \geqslant 0)$. By property (ii) of generalized convolutions it holds also for convex combinations of the measures E_u $(u \geqslant 0)$. Finally, by (iv), it holds for all measures P and Q from $\mathfrak{P}$ because convex combinations of the measures E_u $(u \geqslant 0)$ form a dense subset of $\mathfrak{P}$ in the sense of the weak convergence. Thus the generalized convolution in question is an $(a, 1)$-convolution which completes the proof.

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