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ON THE FIBER-PRESERVING MAPS

BY

JINGYAL PAK (DETROIT, MICHIGAN)

1. Introduction. Let $f\colon X \to X$ be a continuous map on a compact connected ANR X into itself. The Nielsen fixed point theorem says that every map $g\colon X \to X$ homotopic to f has at least $N(f)$ fixed points. Furthermore, in many cases there is a map g homotopic to f and it has exactly $N(f)$ fixed points [11]. Thus the Nielsen fixed point theorem is much more powerful than the Lefschetz fixed point theorem, which only ensures the existence of a single fixed point. The trouble with the Nielsen number is that it is rather hard to compute. In an effort to devise a new tool to compute the Nielsen number of a map, Brown initiated the study of fiber (-preserving) maps in [4].

Let $\mathscr{F} = \{E, \pi, B\}$ be a locally trivial orientable fiber space, where E, B and the fiber $Y \simeq \pi^{-1}(b)$ are connected finite polyhedra. It is well known that the Euler characteristic χ of the spaces involved satisfies $\chi(E) = \chi(B)\cdot\chi(Y)$. Let $f\colon E \to E$ be a fiber (-preserving) map. Then f induces

$$f'\colon B \to B \quad \text{and} \quad f_b\colon \pi^{-1}(b) \to \pi^{-1}(b) \quad \text{for each } b \in B.$$

Brown [3] has shown that the Lefschetz number L of the maps involved satisfies $L(f) = L(f')\cdot L(f_b)$. Note that if f is the identity map, then Brown's result reduces to $\chi(E) = \chi(B)\cdot\chi(Y)$. Unfortunately, similar results for the Nielsen numbers and the fixed point indices do not hold in general.

Let $f\colon E \to E$ be a fiber map and denote the set of fixed points by $\Phi(f)$. Any two points $e_1, e_2 \in \Phi(f)$ are said to be *equivalent* if there are a path C from e_1 to e_2 and a fiber-preserving end points fixed homotopy

$$H\colon I\times I \to E$$

such that, for all $(s,t) \in I\times I$,

$$H(s,0) = C(s), \quad H(s,1) = f\big(C(s)\big) \quad \text{and} \quad \pi H(s,t) = \pi H(s,0).$$

Then $\pi C(s) = \pi f(C(s))$ for all $s \in I$. This is an equivalence relation on $\Phi(f)$ and we can easily see that there are only a finite number of equivalence classes of $\Phi(f)$, since E is compact. It is easily seen that each class belongs to some Nielsen fixed point class of f.

Definition. The number $F(f)$ of a fiber map f is defined to be the total number of the equivalence classes of f which are contained in the essential Nielsen fixed point classes of f.

It is easily seen that $F(f) \geqslant N(f)$. Thus, if $F(f) = 0$, then $N(f) = 0$, and it follows from Lemma 2 that $F(f) \neq 0$ implies $N(f) \neq 0$. Also we infer that if $f, g\colon E \to E$ are fiber-homotopic maps, then $F(f) = F(g)$. Thus $F(f)$ is a useful number in the study of fixed point theorems for the fiber maps.

In an effort to establish a product theorem for the Nielsen numbers, Fadell asks if $F(f) = N(f')\cdot N(f_b)$ is true for each $b \in B$. The following example shows that it is not true in general.

The example concerns the trivial fibration

$$\pi\colon s^2 \times s^1 \to s^2.$$

Let $f'\colon s^2 \to s^2$ be the suspension of a rotation of s^1; then f' has two fixed points, each of local index $+1$. Let $f_b\colon s^1 \to s^1$ be the map of degree -1 obtained by reflecting s^1 about axis; then f_b has two fixed points, each of local index -1 according to a theorem of Leray. We can easily see that $F(f) = 4$ while $N(f') = 1$ and $N(f_b) = 2$. Thus

$$F(f) \neq N(f')\cdot N(f_b).$$

Note that in this example the product theorem holds for the Nielsen number (see [4]), i.e.,

$$N(f) = N(f')\cdot N(f_b) = 2.$$

However, using the $F(f)$ we will show a product theorem which is very useful.

For any map $f\colon X \to X$, let $J(f)$ be the total number of path components contained in the essential Nielsen fixed point classes of f. The purpose of this paper is to show that

$$F(f) = N(f_b)\cdot J(f') \quad \text{for each } b \in B$$

if the spaces involved satisfy the Jiang condition [8].

For the usual definitions and terminologies, readers are referred to [2], [4], and [6].

I am indebted to R. Brown for helpful correspondence.

2. A product theorem. Throughout this section we assume that the spaces involved all satisfy the Jiang condition.

Let $f\colon E \to E$ be a fiber (-preserving) map. Assume that $F^1, \dots, F^n$ and $N^1, \dots, N^m$ are the fixed point classes of f and Nielsen fixed point classes of f', respectively, i.e., $F(f) = n$ and $N(f') = m$. Let N be a Nielsen fixed point class of f such that

$$N \cap \pi^{-1}(N^l) \neq \emptyset,$$

and denote by $\# N$ the number of Nielsen fixed point classes of f_b, $b \in N^l$, in $N \cap \pi^{-1}(b)$. Since $\pi(N) = N^l$ (see [4], p. 493) and the spaces satisfy the Jiang condition, $\# N$ is the invariant of $b \in N^l$. From the definition of the fixed point class of f and from Lemma 5 of [4] we can infer that each class in $N \cap \pi^{-1}(N^l) \neq \emptyset$ contributes exactly one Nielsen class of f_b.

Assume that $L(f) \neq 0$. Then the map $f\colon E \to E$ and the inclusion map $i\colon \pi^{-1}(b) \to E$ induce the following commutative diagram:

$$\begin{tikzcd}
\pi_1\big(\pi^{-1}(b)\big) \arrow[r, "i_\#"] \arrow[d, "1-f_{b\#}"] & \pi_1(E) \arrow[d, "1-f_\#"] \\
\pi_1\big(\pi^{-1}(b)\big) \arrow[r, "i_\#"] & \pi_1(E)
\end{tikzcd}$$

In turn, this diagram induces a homomorphism

$$i^\#\colon \frac{\pi_1\big(\pi^{-1}(b)\big)}{(1-f_{b\#})\pi_1\big(\pi^{-1}(b)\big)} \to \frac{\pi_1(E)}{(1-f_\#)\pi_1(E)}$$

on its cokernels. Let $P(f)$ be the order of $\ker(i^\#)$.

The following lemma is immediate from [10] and from the above remarks.

Lemma 1. *Let $\mathscr{F} = \{E, \pi, B\}$ be an orientable locally trivial fiber space and let $f\colon E \to E$ be a fiber map. Then each essential Nielsen fixed point class N of f contains exactly $k \cdot P(f)$ fixed point classes of f, where k is the number of path components contained in $\pi(N)$. Furthermore, if*

$$N \cap \pi^{-1}(N^l) \neq \emptyset,$$

then $\# N = P(f)$ for each $b \in N^l$.

Note that in $N \cap \pi^{-1}(b)$, $b \in N^l$, the points $e_1, e_2 \in N \cap \pi^{-1}(b)$ are equivalent if and only if they are Nielsen equivalent for f_b. This follows from the equality $f_b = f | \pi^{-1}(b)$ and from Lemma 5 of [4].

Lemma 2. *In Lemma 1, if $b \in N^l$, then $N(f_b) = t \cdot P(f)$, where t is the number of essential Nielsen fixed point classes of f which intersects with $\pi^{-1}(b)$.*

Theorem 1. *Let $\mathscr{F} = \{E, \pi, B\}$ be a locally trivial orientable fiber space, where E, B and the fiber $Y \simeq \pi^{-1}(b)$ are compact connected ANR's.*

Let $f\colon E \to E$ be a fiber map. If the spaces involved satisfy the Jiang condition, then

$$F(f) = J(f')\cdot N(f_b) \quad \textit{for each } b \in B.$$

Proof. Without loss of generality we may assume that $L(f) \neq 0$. By the lemmas, to any essential Nielsen fixed point class N^j of $f'\colon B \to B$ there correspond exactly $N(f_b)$ fixed point classes for each $b \in N^j$. Now, the total number of fixed point classes of f in $\pi^{-1}(N^j)$ is nothing else but $N(f_b)$ times the number of path components of N^j. Since there are altogether $J(f')$ of these components, we can conclude the theorem.

Remark 1. Without the Jiang condition of the spaces involved we have

$$F(f) \leqslant N(f_b)\cdot J(f') \quad \text{for each } b \in \Phi(f').$$

Remark 2. If we can deform f to a fiber-preserving map $g\colon E \to E$ as in Lemma 5 of [4], i.e. we eliminate all the inessential Nielsen fixed point classes of f, then we can state the theorem without the Jiang condition. Readers are referred to [2], Theorem 4 on p. 123, and p. 95.

THEOREM 2. *Assume the hypothesis of Theorem* 1. *If $L(f) \neq 0$, then*

$$\frac{F(f)}{N(f)} = P(f)\cdot\frac{J(f')}{N(f')}.$$

COROLLARY 1. *Let $\mathscr{F} = \{S^{2n+1}, \pi, P^n(C)\}$ be a Hopf fibering $(n \geqslant 1)$ and let $f\colon S^{2n+1} \to S^{2n+1}$ be a fiber-preserving map. Then*

$$F(f) = P(f)\cdot J(f').$$

COROLLARY 2. *Let $\mathscr{F} = \{S^{4n+3}, \pi, P^n(H)\}$ be a quaternionic Hopf fibering $(n \geqslant 1)$ and let $f\colon S^{4n+3} \to S^{4n+3}$ be a fiber-preserving map. Then*

$$F(f) = J(f').$$

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WAYNE STATE UNIVERSITY
DETROIT, MICHIGAN
UNIVERSITY OF HAWAII
HONOLULU, HAWAII

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