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DEDEKIND COMPLETIONS OF LATTICES BY ENDS

BY

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0. Introduction. A generalization of Stone's famous representation theorem was recently published by Leader and Finkelstein working with abstract semilattices [5]. Their main result was the characterization of a certain type of semilattice as a basis for a locally compact Hausdorff space in which each member of the basis is the interior of a compact set. The underlying set on which the topology is defined is the collection of "ends" which may be thought of as generalized ultrafilters.

In this paper, we use these ends to obtain a Dedekind completion of an ordinary lattice. This is accomplished by using the lattice to generate a semilattice of cells similar to those characterized by Leader and Finkelstein. The collection of ends associated with the semilattice can then be given a lattice structure in addition to the topological structure defined by Leader and Finkelstein. The first example shows this procedure to be a generalization of the familiar completion by Dedekind cuts.

In the first section we construct the lattice structure for the collection of ends in order to obtain a Dedekind completion (Theorems 1.1 and 1.2). In the second section we consider these sets as topological spaces noting the relationship between some topological and order properties.

1. Regular lattices.

Definition 1.1. A *regular lattice* $(L, \vee, \wedge, \ll)$ consists of a lattice $(L, \vee, \wedge)$ together with a binary relation $\ll$ defined on L satisfying:

A1. $a \ll b$ implies $a \leqslant b$.

A2. $a \ll b \leqslant c$ or $a \leqslant b \ll c$ implies $a \ll c$.

A3. $a \ll b$ and $c \ll d$ imply $a \vee c \ll b \vee d$ and $a \wedge c \ll b \wedge d$.

A4. If $a \ll c$, then there exists an element b such that $a \ll b \ll c$.

A5. Given b in L, there exist a and c in L such that $a \ll b \ll c$.

A6. If $x \ll a$ implies $x \ll b$, then $a \leqslant b$.

A7. If $a \ll x$ implies $b \ll x$, then $b \leqslant a$.

We shall refer to $(L, \vee, \wedge, \ll)$ as an *R-lattice*. Let a and a' be elements of L with $a \ll a'$. The set $\{x: a \ll x \ll a'\}$ will be denoted by (a, a')

and referred to as a *cell* from L. The collection of all cells from L will be denoted by S.

The proofs of the following propositions are left to the reader.

PROPOSITION 1.1. *If* (a, a') *and* (b, b') *are non-disjoint cells, then*

$$(a, a') \cap (b, b') = (a \vee b, a' \wedge b').$$

PROPOSITION 1.2. *Two cells* (a, a') *and* (b, b') *are non-disjoint if and only if* $a \ll b'$ *and* $b \ll a'$.

PROPOSITION 1.3. $(a, a') \subseteq (b, b')$ *if and only if* $b \leqslant a$ *and* $a' \leqslant b'$.

Example 1.1. Let $(P, \vee, \wedge)$ be any lattice and use the partial ordering $\leqslant$ as the relation $\ll$.

Example 1.2. Let $(R^k, \vee, \wedge)$ be the lattice of k-tuples of real numbers, where $a \vee b$ is the element whose i-th coordinate is the sup of the i-th coordinates of a and b, and $a \wedge b$ is defined with the obvious change. Let $a \ll a'$ mean that the i-th coordinate of a is strictly less than the i-th coordinate of a' for each i. The cell (a, a') would then be the interior of an ordinary k-cell.

For subsets A and B of L, let

$$A \vee B = \{a \vee b \colon a \in A \text{ and } b \in B\}$$

and let $A \ll B$ mean that $a \ll b$ for each a in A and b in B. The obvious change is made for $A \wedge B$. Let A^* and A_* denote the set of upper and lower bounds of A, respectively.

Definition 1.2 (Leader and Finkelstein [5]). Let A be a subset of S. We say that a cell (a, a') *clings to* A if $(a, a') \cap (x, x') \neq \emptyset$ for each (x, x') in A.

Let $\Subset$ be a binary relation defined on S as follows:

$$(a, a') \Subset (b, b') \quad \text{if } b \ll a \text{ and } a' \ll b'.$$

Thus, in Example 1.2, $(a, a') \Subset (b, b')$ means that $\overline{(a, a')} \subseteq (b, b')$ under the usual topology on R^k.

Definition 1.3 (Leader and Finkelstein [5]). For non-empty subsets A and A' of L with $A \ll A'$, let $A \times A'$ denote $\{(a, a') \colon a \in A \text{ and } a' \in A'\}$. The collection of cells $A \times A'$ is called an *end* from S provided that

E1. If (a, a') and (b, b') belong to $A \times A'$, then there exists a cell (c, c') in $A \times A'$ such that $(c, c') \Subset (a, a') \cap (b, b')$.

E2. If (a, a') clings to $A \times A'$ and $(a, a') \Subset (b, b')$, then (b, b') belongs to $A \times A'$.

It follows easily from the definition that each end is maximal with respect to set inclusion. The collection of all ends from S will by denoted by $\boldsymbol{E}$.

The purpose of the next six lemmas will be to construct a lattice structure for $\boldsymbol{E}$.

LEMMA 1.1. *For any two ends $A \times A'$ and $B \times B'$, the following two statements are equivalent*:

(1) *For each a in A, there exists b in B such that $a \ll b$.*

(2) *For each b' in B', there exists a' in A' such that $a' \ll b'$.*

Proof. Assume statement (1) and let $b' \in B'$. By E1 we can choose $x' \ll b'$ with $x' \in B'$. For any a in A, $a \ll x'$. Thus, if $x \in A$ and $(a, a') \in A \times A'$, then $a \ll x'$ and $x \ll a'$, so that (x, x') clings to $A \times A'$. By E2, $b' \in A'$ so, by E1, there exists a' in A' with $a' \ll b'$. The proof of the converse is similar.

Note that statement (1) is equivalent to $A \subseteq B$, and statement (2) means that $B' \subseteq A'$.

LEMMA 1.2. *Let $\leqslant_{\boldsymbol{E}}$ be a binary relation on $\boldsymbol{E}$ defined by either of the statements in Lemma 1.1. Then $\boldsymbol{E}$ is partially ordered by $\leqslant_{\boldsymbol{E}}$.*

Proof. Reflexivity and transitivity are immediate from the definitions and the transitivity of $\ll$. To show antisymmetry assume that

$$A \times A' \leqslant_{\boldsymbol{E}} B \times B' \quad \text{and} \quad B \times B' \leqslant_{\boldsymbol{E}} A \times A'.$$

Let $(a, a') \in A \times A'$. By our assumptions, there exists b in B with $a \ll b$ and there exists b' in B' with $b' \ll a'$. Thus $(b, b') \Subset (a, a')$. Since any member of an end clings to the end, by E2 we have $(a, a') \in B \times B'$. Thus $A \times A' \subseteq B \times B'$. Equality follows from the maximality of ends.

LEMMA 1.3. *Let A be a non-empty subset of L which is bounded from below. Let*

$$B = \{b\colon b \ll x \text{ for some } x \text{ in } A_*\}$$

and let

$$B' = \{b'\colon x \ll b' \text{ for some } x \text{ in } (A_*)^*\}.$$

Then $B \times B'$ is an end.

Proof. Clearly, B and B' are non-empty and $B \ll B'$.

E1. Let (x, x') and (y, y') belong to $B \times B'$. It follows from the definition of $B \times B'$ and from A3 that $x \vee y \in B$ and $x' \wedge y' \in B'$. By A4, there exist z in B and z' in B' with

$$x \vee y \ll z \quad \text{and} \quad z' \ll x' \wedge y',$$

i.e.,

$$(z, z') \Subset (x, x') \cap (y, y').$$

E2. Let (x, x') cling to $B \times B'$ and let $(x, x') \Subset (y, y')$. The proof will be completed if $x \in A^*$ and $x' \in (A_*)^*$. Let $u \in A$ and let v be any element such that $u \ll v$. Then $v \in B'$. Since (x, x') clings to $B \times B'$, we have $x \ll v$. By A7, $x \leqslant u$, so that $x \in A_*$. Similarly, if $u \in A_*$ and $v \ll u$, then $v \in B$, so that $v \ll x'$. Thus $u \leqslant x'$, so $x' \in (A_*)^*$.

We omit the proof of the next lemma due to its similarity to the proof of Lemma 1.3.

LEMMA 1.4. *Let A be a non-empty subset of L which is bounded from above. Let*

$$B' = \{b' \colon x \ll b' \text{ for some } x \text{ in } A^*\}$$

and let

$$B = \{b \colon b \ll x \text{ for some } x \text{ in } (A^*)_*\}.$$

Then $B \times B'$ is an end.

LEMMA 1.5. *Let $A \times A'$ and $B \times B'$ be ends. Let*

$$C = \{c \colon c \ll x \text{ for some } x \text{ in } (A' \vee B')_*\}$$

and let

$$C' = \{c' \colon x \ll c' \text{ for some } x \text{ in } ((A' \vee B')_*)^*\}.$$

Then $C \times C'$ is the l.u.b. of $A \times A'$ and $B \times B'$ under the partial ordering $\leqslant_E$.

Proof. Since $A' \vee B'$ is bounded from below, $C \times C'$ is an end by Lemma 1.3. Let $x \in A$ and choose a in A with $x \ll a$. Then, for any b in B, $x \ll a \leqslant a \vee b$. Since $a \vee b \in C$, we have

$$A \times A' \leqslant_E C \times C'.$$

Similarly, $B \times B' \leqslant_E C \times C'$. Suppose that $D \times D'$ is also an upper bound. Then, for any d' in D', there exist a' in A' and b' in B' such that $a' \vee b' \ll d'$. Since $a' \vee b' \in C'$, we have

$$C \times C' \leqslant_E D \times D'.$$

Once again we omit the proof of the lemma that follows because of its similarity to the proof of Lemma 1.5.

LEMMA 1.6. *Let $A \times A'$ and $B \times B'$ be ends. Let*

$$C' = \{c' \colon x \ll c' \text{ for some } x \text{ in } (A \wedge B)^*\}$$

and let

$$C = \{c \colon c \ll x \text{ for some } x \text{ in } ((A \wedge B)^*)_*\}.$$

Then $C \times C'$ is the g.l.b. of $A \times A'$ and $B \times B'$ under $\leqslant_E$.

We shall use the symbols $\vee_E$ and $\wedge_E$ for l.u.b. and g.l.b., respectively, under the partial ordering $\leqslant_E$.

THEOREM 1.1. *$(E, \vee_E, \wedge_E)$ is a Dedekind complete lattice.*

Proof. The preceding lemmas establish the lattice structure. Let Λ be some index set and assume that $\{A_\alpha \times A'_\alpha\}$ $(\alpha \in \Lambda)$ is a collection of ends bounded from above by $A \times A'$. Let

$$B' = \Big\{b' \colon x \ll b' \text{ for some } x \text{ in } \Big(\bigcup_{\alpha\in\Lambda} A_\alpha\Big)^*\Big\}$$

and let

$$B = \Big\{b \colon b \ll x \text{ for some } x \text{ in } \Big(\Big(\bigcup_{\alpha\in\Lambda} A_\alpha\Big)^*\Big)_*\Big\}.$$

Since $\bigcup_{\alpha\in\Lambda} A_\alpha$ is non-empty and bounded from above by any element in A', we infer from Lemma 1.4 that $B \times B'$ is an end. Also, for any $\alpha \in \Lambda$, $A_\alpha \subseteq B$, so that $B \times B'$ is an upper bound for the collection. To show that $B \times B'$ is the l.u.b., assume that $C \times C'$ is an upper bound and $C \times C' \leqslant_E B \times B'$. For any (c, c') in $C \times C'$, there exists b in B such that $c \ll b$. Thus $c \in B$. Also, since $C \times C'$ is an upper bound of $\{A_\alpha \times A'_\alpha\}$ $(\alpha \in \Lambda)$, we have

$$C' \subseteq \Big(\bigcup_{\alpha\in\Lambda} A_\alpha\Big)^*,$$

so that $C' \subseteq B'$. Thus, $(c, c') \in B \times B'$ from which we obtain $B \times B' = C \times C'$.

We proceed to embed the R-lattice $(L, \vee, \wedge)$ into $(E, \vee_E, \wedge_E)$. This is done in the natural manner suggested by the following

LEMMA 1.7. *Let $x \in L$ and let $\mathcal{N}_x = \{(y, y') \colon x \in (y, y')\}$. Then $\mathcal{N}_x$ is an end.*

Proof. Clearly, $\mathcal{N}_x$ is of the form $A \times A'$, where $A = \{a \colon a \ll x\}$ and $A' = \{a' \colon x \ll a'\}$. Let (a, a') and (b, b') belong to $\mathcal{N}_x$. By A3,

$$a \vee b \ll x \ll a' \wedge b'$$

and, by A4, there exist c and c' such that

$$a \vee b \ll c \ll x \ll c' \ll a' \wedge b'.$$

Thus E1 is satisfied.

Let (p, p') cling to $\mathcal{N}_x$ and $(p, p') \Subset (q, q')$. If $x \notin (q, q')$, then either q non$\ll x$ or x non$\ll q'$. Assuming the former, we have $p \not\ll x$. By A7, there exists y' such that $x \ll y'$ and p non$\ll y'$. Thus, for any $y \ll x$, $(y, y') \in \mathcal{N}_x$ but

$$(y, y') \cap (p, p') = \emptyset,$$

contradicting that (p, p') clings to $\mathcal{N}_x$. A similar contradiction is obtained by assuming x non$\ll q'$.

THEOREM 1.2. *The mapping $f(x) = \mathcal{N}_x$ is an isomorphism from $(L, \vee, \wedge)$ into $(E, \vee_E, \wedge_E)$. Furthermore, the mapping is onto if and only if L is Dedekind complete.*

Proof. If $\mathcal{N}_x = \mathcal{N}_y$, then, by A6 and A7, $x = y$.

Let $f(a) = A \times A'$ and $f(b) = B \times B'$. We shall show that

$$f(a \vee b) \subseteq A \times A' \vee_E B \times B'$$

from which equality follows by the maximality of ends. Let $(x, x') \in f(a \vee b)$ and choose (y, y') in $f(a \vee b)$ with $(y, y') \Subset (x, x')$. Since $y \ll a \vee b$, we have $y \in (A' \vee B')_*$. Similarly, since $a \vee b \ll y'$, we obtain $y' \in \big((A' \vee B')_*\big)^*$. Thus, (x, x') belongs to $A \times A' \vee_E B \times B'$. The preservation of the meet operation is proved in a similar manner.

To show the second part of the theorem, assume that $\langle L, \vee, \wedge\rangle$ is Dedekind complete and that $A \times A'$ is an end. Since each element of A' is an upper bound of A, there exists an l.u.b. x of A and $a \leqslant x \leqslant a'$ for each (a, a') in $A \times A'$. By E1, this can be strengthened to $a \ll x \ll a'$. Hence, $A \times A' \subseteq \mathcal{N}_x$, so, by the maximality of ends, $A \times A' = \mathcal{N}_x$.

If the mapping is onto, then L is Dedekind complete by Theorem 1.1.

2. Topological structure of R-lattices and ends.

LEMMA 2.1. *Let a and b be distinct elements of L. Then there exist disjoint cells (x, x') and (y, y') containing a and b, respectively.*

Proof. Assume $a \not\leqslant b$. Then there exists $z \ll a$ with z non$\ll b$. Choose z' such that $a \ll z'$. Then $a \in (z, z')$ and $b \notin (z, z')$. Choose (x, x') containing a with $(x, x') \Subset (z, z')$. Then $x \not\leqslant b$, so there exists an element y' such that $b \ll y'$ and x non $\ll y'$. Choosing any $y \ll b$, we have $b \in (y, y')$ and $(x, x') \cap (y, y') = \emptyset$.

THEOREM 2.1. *The collection $\boldsymbol{B} = \{(a, a'): (a, a') \in S\}$ is a base for a T_2-topology on L in which*

$$\overline{(a, a')} = \{x: a \leqslant x \leqslant a'\}.$$

Proof. The fact that $\boldsymbol{B}$ is a base for a T_2-topology follows from Proposition 1.1 and Lemma 2.1. Suppose that $x \in \overline{(a, a')}$. If $y \ll x$, then, for any y' such that $x \ll y'$, we have

$$(y, y') \cap (a, a') \neq \emptyset,$$

so that $y \ll a'$. Thus, by A6, $x \leqslant a'$. Similarly, $x \ll y'$ implies $a \ll y'$, so that, by A7, $a \leqslant x$. If $x \notin \overline{(a, a')}$, then there exists a cell (y, y') containing x such that

$$(y, y') \cap (a, a') = \emptyset.$$

If $a \leqslant x \leqslant a'$, then $y \ll a'$ and $a \ll y'$ contradicting Proposition 1.2.

PROPOSITION 2.1. *If $\{x: a \leqslant x \leqslant a'\}$ is compact for each cell (a, a'), then $\langle L, \vee, \wedge\rangle$ is Dedekind complete.*

Proof. Suppose that $(L, \vee, \wedge)$ is not Dedekind complete. Then by Theorem 1.2 there exists an end $A \times A'$ which is not of the form $\mathscr{N}_x$. Choose any $(a, a') \in A \times A'$. Consider the collection $\{L - \{x\colon u \leqslant x \leqslant u'\}\}$ for all (u, u') in $A \times A'$. Since $A \times A'$ is not of the form $\mathscr{N}_x$, for each y in L there exists a cell (u, u') in $A \times A'$ such that $y \notin (u, u')$. Then choosing (v, v') in $A \times A'$ with $(v, v') \Subset (u, u')$, we have $y \notin \{x\colon v \leqslant x \leqslant v'\}$. Thus, the collection is an open cover of L, hence of $\{x\colon a \leqslant x \leqslant a'\}$. Choose any finite subcollection

$$\{L - \{x\colon u_i \leqslant x \leqslant u_i'\}\} \quad (i = 1, 2, \ldots, n).$$

Let

$$u = \text{l.u.b.}\{u_i\} \quad \text{and} \quad u' = \text{g.l.b.}\{u_i'\} \quad (i = 1, 2, \ldots, n).$$

Clearly, no element of (u, u') is contained in any member of the subcollection. Since $(u, u') \in A \times A'$, we have

$$(u, u') \cap (a, a') \neq \emptyset,$$

so that $\{x\colon a \leqslant x \leqslant a'\}$ is not covered by the subcollection and is non--compact.

For each cell (a, a'), let $[(a, a')]$ denote the collection of ends to which (a, a') belongs. Then, since

$$[(a, a')] \cap [(b, b')] = [(a, a') \cap (b, b')] = [(a \vee b, a' \wedge b')],$$

it follows from Lemma 2.1 that $\{[(a, a')]\colon (a, a') \in S\}$ is a base for a T_2--topology on $\boldsymbol{E}$.

THEOREM 2.2. *The mapping from L to $\boldsymbol{E}$ defined by $f(x) = \mathscr{N}_x$ is a homeomorphism onto a dense subset of $\boldsymbol{E}$.*

Proof. The fact that f is one-to-one was established in Theorem 1.2. It is clear that both f and f^{-1} are continuous, since

$$f\big((a, a')\big) = f(L) \cap [(a, a')],$$

i.e., the topology defined on L is the topology it inherits as a subspace of $\boldsymbol{E}$. To show that $f(L)$ is dense in $\boldsymbol{E}$, let $A \times A'$ be any end, and $[(a, a')]$ a basic open set containing $A \times A'$. Then $(a, a') \in A \times A'$ and $\mathscr{N}_x \in [(a, a')]$ for any $x \in (a, a')$.

PROPOSITION 2.2. *If $\boldsymbol{E}$ is compact, then $(L, \vee, \wedge)$ is bounded.*

Proof. By compactness, $\boldsymbol{E}$ is covered by a finite collection $\{[(a_i, a_i')]\}$ $(i = 1, 2, \ldots, n)$. Let

$$b = \text{g.l.b.}\{a_i\} \quad \text{and} \quad b' = \text{l.u.b.}\{a_i'\} \quad (i = 1, 2, \ldots, n).$$

Then $\boldsymbol{E} = [(b, b')]$. Thus, for any a in L, $\mathscr{N}_a \in [(b, b')]$, so $b \ll a \ll b'$.

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