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SPECTRAL RADIUS AND SEMINORMS IN FINITE-DIMENSIONAL ALGEBRAS. II

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In this paper the authors consider conditions for a seminorm p on an algebra $\mathscr{A}$ to satisfy $\lim p(A^k)^{1/k} = r(A)$ for all $A \in \mathscr{A}$, where $r(A)$ is the spectral radius of A. This paper extends previous work of the second-named author [4].

Let $r(A) = \max\{|\lambda|: \lambda \text{ an eigenvalue of } A\}$ be the *spectral radius* of $A \in M_n(C)$. In [5], Wimmer has shown

$$r(A) = \overline{\lim}\, |\operatorname{tr}(A^k)|^{1/k} \quad \text{for all } A \in M_n(C)$$

and raised the question of characterizing those seminorms p on $M_n(C)$ for which

$$r(A) = \lim p(A^k)^{1/k} \quad \text{for all } A \in M_n(C).$$

In [4], the second-named author has shown

$$\overline{\lim}\, p(A^k)^{1/k} = r(A) \quad \text{for all } A \in M_n(C)$$

iff $\ker(p)$ contains no non-zero idempotent matrix, where $\ker(p) = \{A: p(A) = 0\}$ is a subspace of $M_n(C)$.

In the present paper, the authors consider necessary and sufficient conditions for a seminorm p to satisfy $\lim p(A^k)^{1/k} = r(A)$ for all $A \in \mathscr{A}$, where $\mathscr{A}$ is a subalgebra of $M_n(C)$. It should be mentioned that any n-dimensional complex Banach algebra with identity may be regarded as a subalgebra of $M_n(C)$ by considering the left or right regular representation of the algebra.

In view of the results in [4], it is natural to consider the following statement in which p is an arbitrary seminorm on a finite-dimensional complex algebra $\mathscr{A}$:

(1) $r(A) = \lim p(A^k)^{1/k}$ for all $A \in \mathscr{A}$ iff $A \in \mathscr{A}$, $0 \neq A = A^m$, for some $m > 1$ implies $A \notin \ker(p)$.

We will show that (1) is false in general, and obtain a characterization of those algebras $\mathscr{A}$ for which (1) holds.

To show that (1) does not hold in general, consider the following

Counterexample. Let $\{m_k\}$ be any sequence of integers satisfying $m_{k+1} \geqslant 2^{m_k} + m_k + 1$ and let

$$w = \sum_{k=1}^{\infty} 2^{-m_k}.$$

Note that w is irrational since its binary expansion is non-repeating. Therefore $\alpha = \exp(2\pi i w)$ is not an integral root of 1.

If p is defined on $M_2(C)$ by

$$p\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = |b| + |c| + |a - \alpha d|,$$

then p is a seminorm whose kernel consists of multiples of

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix},$$

and so it is clear that $\ker(p)$ contains no non-zero B satisfying $B^m = B$ for some $m > 1$. We also note that $r(A) = 1$ and claim that

$$\underline{\lim}\, p(A^k)^{1/k} \leqslant 1/2.$$

To see this, note that

$$\begin{aligned} p(A^{2^{m_r}+1}) &= |\alpha^{2^{m_r}+1} - \alpha| = |\alpha^{2^{m_r}} - 1| \\ &= \left|\exp\left(2\pi i \sum_{k=1}^{\infty} 2^{m_r - m_k}\right) - 1\right| = \left|\exp\left(2\pi i \sum_{k=r+1}^{\infty} 2^{m_r - m_k}\right) - 1\right| \\ &\leqslant 2\pi \sum_{k=r+1}^{\infty} 2^{m_r - m_k} \leqslant 2\pi \cdot 2^{m_r} \sum_{k=m_{r+1}}^{\infty} 2^{-k} \\ &= 4\pi \cdot 2^{m_r - m_{r+1}} \leqslant 4\pi \cdot 2^{-(2^{m_r}+1)}. \end{aligned}$$

Now raise both ends of the inequality to the power $(2^{m_r}+1)^{-1}$ and let $r \to \infty$. This establishes the claim and the counterexample.

THEOREM. *Suppose $\mathscr{A}$ is a finite-dimensional complex algebra with identity I. The following are equivalent:*

(a) (1) *holds*;

(b) $\lim p(A^k)^{1/k} = r(A)$ *for all* $A \in \mathscr{A}$ *and for every seminorm* p *for which* $p(I) > 0$;

(c) *for every* $A \in \mathscr{A}$, *the minimal polynomial of* A *has only one root*;

(d) *for some* n, $\mathscr{A}$ *is the algebra isomorphic to a subalgebra of the* n-*by*-n *upper triangular matrices with constant main diagonal.*

Proof. (a) $\Rightarrow$ (c). If (c) does not hold, then $\mathscr{A}$ contains a subalgebra $\mathscr{A}_1$ which is isomorphic to

$$\frac{C[x]}{\langle (x-\alpha)^m \rangle} \times \frac{C[x]}{\langle (x-\beta)^t \rangle}$$

for some $\alpha \neq \beta$ and positive integers m, t; that is, $\mathscr{A}_1$ is isomorphic to the algebra generated by a "two-block" Jordan matrix. After the fashion of the counterexample preceding the Theorem one can produce a seminorm p on $\mathscr{A}_1$ so that $\ker(p)$ contains no non-zero A such that $A = A^m$ for some $m > 1$, but for which $\underline{\lim}\, p(A^k)^{1/k} < r(A)$ for some $A \in \mathscr{A}_1$. We may extend p to $\mathscr{A}$ in a way such that not to disturb $\ker(p)$ to see that (a) fails.

(c) $\Rightarrow$ (b). Suppose (c) holds, p is a seminorm on $\mathscr{A}$ for which $p(I) > 0$, and $A \in \mathscr{A}$. Without loss of generality we may assume that A is not a scalar and A is not nilpotent, so that $A = \alpha I + N$, where $0 \neq \alpha \in C$ and N is nilpotent. If $\mathscr{A}_1$ is the algebra generated by I and N, then $\mathscr{A}_1$ has the basis $I, N, N^2, \ldots, N^{m-1}$ (with $N^m = 0$). Since $I \in \mathscr{A}_1$, $\mathscr{A}_1$ is not contained in $\ker(p)$, so there is a non-zero linear functional $f \in \mathscr{A}_1^*$ which annihilates $\mathscr{A}_1 \cap \ker(p)$. Then f can be represented by a sequence $(b_0, \ldots, b_{m-1})$ as follows:

$$f\Big(\sum_{i=0}^{m-1} c_i N^i\Big) = \sum_{i=0}^{m-1} c_i b_i .$$

Since f annihilates $\mathscr{A}_1 \cap \ker(p)$, p dominates (see [4]) the seminorm q defined on $\mathscr{A}_1$ by $q(B) = |f(B)|$. It is straightforward to see that

$$\lim q(A^k)^{1/k} = |\alpha| = r(A),$$

since the b_k are not all zero, whence

$$\underline{\lim}\, p(A^k)^{1/k} \geqslant r(A),$$

which suffices to establish (b) (see [4]).

(c) $\Leftrightarrow$ (d). Clearly (d) $\Rightarrow$ (c). For the converse, note that every $A \in \mathscr{A}$ is of the form $\alpha I + N$, where N is nilpotent, and that $\mathscr{A} = \langle I \rangle + \mathscr{A}_0$, where $\mathscr{A}_0 = \{A \in \mathscr{A}\colon A \text{ is nilpotent}\}$. Since $\mathscr{A}_0$ is a finite-dimensional nilalgebra, results of Gerstenhaber [1]-[3] yield that $\mathscr{A}_0$ is isomorphic to an algebra of strictly upper triangular matrices, which surely implies (d).

(b) $\Rightarrow$ (a). This implication follows since the "only if" part of (1) holds for any $\mathscr{A}$ and p. This completes the proof of the Theorem.

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