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ON ABEL SUMMABILITY OF MULTIPLE JACOBI SERIES

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Introduction. The purpose of this paper is to extend to the case of Jacobi series the results of [4] and [3] as well as those of [5] concerning Abel summability of ultraspherical series.

Besides the pointwise a. e. convergence of the restricted Abel summability of a multiple Jacobi series of a function belonging to L^1, the weak type 1-1 estimate for the corresponding maximal operator is proved. The latter is the main part of this paper. The results of [2] play an important role in getting the weak-type inequality for the "max" function.

0. Some notation. We shall be concerned throughout this paper with the spaces $L^p(J^{\alpha,\beta})$ of measurable functions defined on the unit cube Q of R^k.

So the $L^p(J^{\alpha,\beta})$-norm of f will be defined as

$$\|f\|^p_{p(J^{\alpha,\beta})} = \int_Q |f|^p(1-Y)^\alpha(1+Y)^\beta dY \tag{0.1}$$

$$= \int_Q |f|^p \prod_{j=1}^k (1-y_j)^{\alpha_j}(1+y_j)^{\beta_j} dy_1 \ldots dy_k,$$

where $\alpha_j > -1$, $\beta_j > -1$, $j = 1, 2, \ldots, k$, $p \geqslant 1$. We shall also denote the integral with respect to the measure $(1-Y)^\alpha(1+Y)^\beta dY$ in the following way:

$$\int_Q f(1-Y)^\alpha(1+Y)^\beta dY = \int_Q f dJ^{\alpha,\beta}. \tag{0.2}$$

The $(1-Y)^\alpha(1+Y)^\beta dY$-measure of the set, where $f \geqslant \lambda$ $(f \geqslant 0)$ and $\lambda \geqslant 0$, will be denoted as

$$J^{\alpha,\beta}\{X;\ f(X) \geqslant \lambda\}. \tag{0.3}$$

$P_n^{\alpha,\beta}$ will denote the n-th normalized Jacobi polynomial of parameters α, β.

Given a measurable function $f \in L^1(J^{a,\beta})$, we define its multiple Fourier-Jacobi coefficients in the following way:

$$(0.4) \qquad C^{a,\beta}_{n_1,n_2,\ldots,n_k}(f) = C_{n_1,\ldots,n_k} = \int_Q f\Big(\prod_{j=1}^k P^{a_j\beta_j}_{n_j}\Big)\,dJ^{a,\beta}.$$

1. Statement of results. Let $K^{a,\beta}(r, X, Y)$ denote the k-dimensional Watson kernel defined by

$$(1.1.1) \quad K^{a,\beta}(r, X, Y) = \prod_{j=1}^k K^{a_j\beta_j}(r_j, x_j, y_j) = \prod_{j=1}^k \Big(\sum_{n=0}^\infty r_j^n P^{(a_j,\beta_j)}_n(x_j)P^{(a_j,\beta_j)}_n(y_j)\Big),$$

where $0 < r_j < 1$.

By using the estimates of [6], p. 67 and p. 163, we get

$$(1.1.2)' \qquad |P^{(a,\beta)}_n(x)| \leqslant C\cdot n^{q+1/2},$$

where $q = \max(a, \beta) \geqslant -1/2$. Consequently, we have

$$(1.1.3) \qquad \begin{aligned} f(r, X) &= \sum C_{n_1,\ldots,n_k}\cdot r_1^{n_1}\ldots r_k^{n_k}\cdot P^{(a_1,\beta_1)}_{n_1}(x_1)\ldots P^{(a_k,\beta_k)}_{n_k}(x_k) \\ &= \int_Q K^{a,\beta}(r, X, Y)f(Y)(1-Y)^a(1+y)^\beta dY, \end{aligned}$$

where $f \in L^1(J^{a,\beta})$.

Notice that (1.1.2) implies

$$(1.1.4) \qquad |C_{n_1,\ldots,n_k}| \leqslant \Big(\prod_1^k n_j^{q+1/2}\Big)\|f\|_{1(J^{a,\beta})}\cdot C,$$

where $\|f\|_{p(J^{a,\beta})}$ denotes the $L^p(J^{a,\beta})$-norm of f.

1.2. THEOREM. *Let $a_i, \beta_i > -1$, $a_i+\beta_i > -1$ and*

$$\vec{r}(t)\colon I \to \mathop{\mathsf{X}}_{s=1}^k I_s$$

be an increasing and continuous function on each component such that $\vec{r}(0) = 0$ and $r(1) = (1, \ldots, 1)$, where $I = I_s = [0, 1]$. Let us define, for $f \in L^1(J^{a,\beta})$,

$$Mf(X) = \sup_{0<t<1} |f(r(t), X)|.$$

Then

(i) *if $f \in L^p(J^{a,\beta})$, $1 < p \leqslant \infty$, then $Mf(X) \in L^p(J^{a,\beta})$ and $\|Mf(X)\|_{p(J^{a,\beta})} \leqslant C(p)\|f\|_{p(J^{a,\beta})}$;*

(ii) *if $f \in L^1(J^{a,\beta})$, then*

$$J^{a,\beta}\{X;\ Mf(X) > \lambda\} \leqslant \frac{C}{\lambda}\|f\|_{1(J^{a,\beta})}.$$

1.3. THEOREM. *Under the conditions on* α, β *and* $\vec{r}(t)$ *of Theorem* 1.2, *if* $f \in L^p(J^{\alpha,\beta})$ $(1 \leqslant p < \infty)$, *then*

$$\lim_{t\to 1} f(r, X) = f(X)$$

a. e. and in the L^p*-norm.*

2. Auxiliary lemma.

(2.1) LEMMA 1. *Let* S *be a bounded subset of* R^m *such that for each* x *belonging to* S *there exists an* m*-dimensional rectangle* $R(x)$, *centered at* x, *such that:*

(a) *the edges of* $R(x)$ *are parallel to the coordinate axes;*

(b) *the length of the edge of* $R(x)$ *corresponding to the* j*-th axis is given by*

$$K_j\varphi_j^{1/2}(t)\{h_j(x_j)+\varphi_j(t)\}^{1/2}, \tag{2.1.1}$$

where $t = t(x)$ *and* h_j *is a function depending on* x_j *only, verifying the Lipschitz condition*

$$|h_j(s_1)-h_j(s_2)| < C_j|s_1-s_2|, \quad j = 1, 2, \ldots, m, \ C_j > 0. \tag{2.1.2}$$

The $\varphi_j(t)$, $j = 1, 2, \ldots, m$, *are increasing functions of the parameter* $t \geqslant 0$, *continuous at* $t = 0$, $\varphi_j(0) = 0$.

Under the preceding assumptions, it is possible to select a subsequence $\{R(x_n)\}$ *of rectangles satisfying*

(i) $S \subset \bigcup_1^\infty R(x_n)$;

(ii) *each* $x \in R^m$ *belongs to at most*

$$C(m)\prod_{j=1}^{m}[1+\log_2(1+C_jK_j)] \tag{2.1.3}$$

different rectangles.

Proof [1]. We may assume that $K_j = 1$, since the general case can be reduced to this case.

Let us decompose S in the following way:

$$S = \bigcup_{l_1,\ldots,l_m} S_{l_1,\ldots,l_m}, \quad \text{where } l_i = 0 \text{ or } 1. \tag{2.1.4}$$

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[1] The technique employed in this proof can be used in other type of situations than that of (2.1.1). For example, if instead of the function of (2.1.1) we have $K_j\varphi_j^{1/2}(f)P^a(h_j, \varphi_j)$, where $a > 0$ and P is a homogeneous polynomial of degree n with coefficient 1 in the term x^n.

Since for each $x \in S$ there exists a rectangle $R(x, t(x))$ whose center is x, it is possible to define a partition of S according to the following

$$(2.1.5)\qquad l_i = \begin{cases} 0 \Leftrightarrow \varphi_i(t(x)) \leqslant h_i(x_i)\, \dfrac{1}{2^8[C_i^2+1]}, \\ 1 \Leftrightarrow \varphi_i(t(x)) > h_i(x_i)\, \dfrac{1}{2^8[C_i^2+1]}. \end{cases}$$

Then x belongs to $S_{l_1,\ldots,l_m}$ if and only if (2.1.5) is verified. Note that there are at most 2^m different sets.

Let us consider a given $S_{l_1,\ldots,l_m}$. Without loss of generality we may assume that

$$(2.1.6)\qquad l_i = \begin{cases} 0 & \text{for } 0 \leqslant i \leqslant k, \\ 1 & \text{for } k < i \leqslant m. \end{cases}$$

Our next step will be to define a partition of the space, where each $S_{l_1,\ldots,l_m}$ will be given by the union

$$(2.1.7)\qquad S_{l_1,\ldots,l_m} = \bigcup_{d_1,\ldots,d_k} S^{d_1,\ldots,d_k}_{l_1,l_2,\ldots,l_m},$$

where $d_i \in Z$ (the set of integers).

We shall say that x belongs to $S^{d_1,\ldots,d_k}_{l_1,\ldots,l_m}$ if $x \in S_{l_1,\ldots,l_m}$ and if

$$(2.1.8)\qquad \{2^{d_i-1} < h_i(x_i) \leqslant 2^{d_i}\} \quad \text{for } 1 \leqslant i \leqslant k \ (d_i \in Z).$$

We shall show that on each $S^{d_1,\ldots,d_k}_{l_1,\ldots,l_m}$ we are in condition to apply lemma 3 of [2]. In fact, let x and y be points of $S^{d_1,\ldots,d_k}_{l_1,\ldots,l_m}$ and suppose

$$(2.1.9)\qquad t(x) = t_1 \leqslant t_2 = t(y).$$

Then for $i \leqslant k$ we have

$$(2.1.10)\qquad \begin{aligned} \varphi_i^{1/2}(t_1)\{\varphi_i(t_1)+h_i(x_i)\}^{1/2} &\leqslant \varphi_i^{1/2}(t_2)\{\varphi_i(t_2)+h_i(x_i)\}^{1/2} \\ &\leqslant 2^{1/2}\varphi_i^{1/2}(t_2)\{\varphi_i(t_2)+h_i(y_i)\}^{1/2}. \end{aligned}$$

This inequality follows from the fact that $h_i(x_i) \leqslant 2h_i(y_i)$ (see (2.1.8)). If $i > k$, we have

$$(2.1.11)\qquad \varphi_i^{1/2}(t_1)\{\varphi_i(t_1)+h_i(x_i)\}^{1/2} \leqslant \varphi_i^{1/2}(t_2)\{\varphi_i(t_2)+h_i(x_i)\}^{1/2}.$$

Notice that

$$(2.1.12)\qquad h_i(x_i) \leqslant 2^8[C_i^2+1]\varphi_i(t_1) \leqslant 2^8[C_i^2+1]\varphi_i(t_2).$$

We translate this estimate to (2.1.11) and obtain

$$(2.1.13)\qquad \begin{aligned} \varphi_i^{1/2}(t_1)\{\varphi_i(t_1)+h_i(x_i)\}^{1/2} &\leqslant 2^5[C_i^2+1]^{1/2}\varphi_i(t_2) \\ &\leqslant 2^5[C_i^2+1]\varphi_i^{1/2}(t_2)[\varphi_i(t_2)+h_i(y_i)]^{1/2}. \end{aligned}$$

Now, we may apply lemma 3 of [2] and get for $S^{d_1,\dots,d_k}_{l_1,\dots,l_m}$ a subcovering (at most denumerable) such that each point of R^m belongs to at most

$$2^m m!\,2^k \prod_{j=k+1}^{m} [1+\log_2\{1+2^5(C_j^2+1)^{1/2}\}]. \tag{2.1.14}$$

different rectangles. Instead of the preceding estimate we shall use a bigger one, namely

$$2^m m! \prod_{j=1}^{m} \{1+\log_2[1+2^5(C_j^2+1)^{1/2}]\}. \tag{2.1.15}$$

Suppose now that $x \in S^{d_1,\dots,d_k}_{l_1,\dots,l_m}$ and y belongs to $R(x)$. Then

$$|x_i-y_i| \leqslant \tfrac{1}{2}\varphi_i^{1/2}(t_x)\{\varphi_i(t_x)+h_i(x_i)\}^{1/2} \tag{2.1.16}$$

and we have

$$|h_i(x_i)-h_i(y_i)| < C_i|x_i-y_i| \leqslant \tfrac{1}{2}C_i\varphi_i^{1/2}(t_x)\{\varphi_i(t_x)+h_i(x_i)]^{1/2}. \tag{2.1.17}$$

If $i \leqslant k$, the last inequality is dominated by

$$\frac{\frac{1}{2}C_i h_i(x_i)}{2^4[C_i^2+1]^{1/2}}\left[1+\frac{1}{2^8(C_i^2+1)}\right]^{1/2} \leqslant 2^{d_i-3}, \tag{2.1.18}$$

which follows from (2.1.5). Thus

$$2^{d_i-2} \leqslant h_i(y_i) \leqslant 2^{d_i+1}, \qquad 0 \leqslant i \leqslant k. \tag{2.1.19}$$

Recalling that $(l_1, \dots, l_m)$ has been fixed, suppose now that we are given a y that verifies

$$2^{d_i-1} < h_i(y_i) \leqslant 2^{d_i}, \qquad i = 1, 2, \dots, m.$$

Suppose also that y belongs to some $R(x)$ with x belonging to $S_{l_1,\dots,l_m}$. Then it follows from (1.2.16) to (1.2.19) that x belongs to some $S^{d_1+\varepsilon_1,d_2+\varepsilon_2,\dots,d_k+\varepsilon_k}_{l_1,\dots,l_m}$ with ε_i equal to 1, 0 or -1; that is, x belongs to at most 3^k different sets $S^{d_1,\dots,d_k}_{l_1,\dots,l_m}$. Since for each $d_1, \dots, d_k$ we have defined a covering, we infer that y belongs to at most 3^k coverings.

So y belongs to at most

$$3^k 2^m m! \prod_{j=1}^{m} \{1+\log_2[1+2^5(C_j^2+1)^{1/2}]\} \tag{2.1.20}$$

different rectangles centered at $S_{l_1,\dots,l_m}$. Since there are at most 2^m sets $S_{l_1,\dots,l_m}$, it will mean that y belongs to at most

$$12^m m! \prod_{j=1}^{m} \{1+\log_2[1+2^5(C_j^2+1)^{1/2}]\} \tag{2.1.21}$$

different rectangles. This completes the proof.

3. Estimates for the Watson kernel. The single Watson kernel takes the form

(3.1.1)

$$K^{a,\beta}(r,x,y) = r^{1/2-a/2-\beta/2}\frac{d}{dr}\left\{k^{1+2+\beta}\int_0^{\pi/2}\frac{1}{Z_1^a Z_2^\beta Y}\sec^{a+\beta+2} w\cos(a-\beta)w\,dw\right\},$$

where

$$0 \leqslant w \leqslant \pi/2, \qquad k = \tfrac{1}{2}(r^{1/2}+r^{-1/2}),$$

$$u = \tfrac{1}{2}(1-x)^{1/2}(1-y)^{1/2}, \qquad v = \tfrac{1}{2}(1+x)^{1/2}(1+y)^{1/2},$$

$$Y = \{[(k\sec w)^2-u^2-v^2]^2-4u^2v^2\}^{1/2},$$

$$Z_1 = (k\sec w)^2+u^2-v^2+Y, \qquad Z_2 = (k\sec w)^2-u^2+v^2+Y.$$

This formula can be found in [1], p. 272. We use the alternative form:

$$Y^2 = \left(\frac{x-y}{2}\right)^2+(k^2\sec^2 w-1)(k^2\sec^2 w-xy),$$

$$Z_1 = k^2\sec^2 w-\tfrac{1}{2}(x+y)+Y, \qquad Z_2 = k^2\sec^2 w+\tfrac{1}{2}(x+y)+Y.$$

We shall decompose the single kernel in the sum of four kernels A, B, C, D defined in the following way:

$$\text{(3.2.1)}\quad A = t^{1/2-a/2-\beta/2}\frac{d}{dt}\{k^{1+a+\beta}\}\int_0^{\pi/2}\frac{1}{Z_1^a Z_2^\beta Y}\sec^{2+a+\beta} w\cos(a-\beta)w\,dw,$$

$$\text{(3.2.2)}\quad B = t^{1/2-a/2-\beta/2}k^{1+a+\beta}\int_0^{\pi/2}\frac{d}{dt}(Y^{-1})\frac{\sec^{2+a+\beta} w}{Z_1^a Z_2^\beta}\cos(a-\beta)w\,dw,$$

$$\text{(3.2.3)}\quad C = t^{1/2-a/2-\beta/2}k^{1+a+\beta}\int_0^{\pi/2}\frac{d}{dt}(Z_1^{-a})\frac{\sec^{2+a+\beta} w}{Z_2^\beta Y}\cos(a-\beta)w\,dw,$$

$$\text{(3.2.4)}\quad D = t^{1/2-a/2-\beta/2}k^{1+a+\beta}\int_0^{\pi/2}\frac{d}{dt}(Z_2^{-\beta})\frac{\sec^{2+a+\beta} w}{Z_1^a Y}\cos(a-\beta)w\,dw.$$

Before beginning with the study of the kernels, we shall state some elementary estimates.

Let $k\sec w = s$, $1 \leqslant s \leqslant 2$, $|y| \leqslant 1$, $0 \leqslant x \leqslant 1$. Then we have

$$\text{(3.2.5)}\quad s^2-\min(x,y) \leqslant 4[s-\min(x,y)] \leqslant 8[s-xy] \leqslant 16[s-\min(x,y)] \leqslant 16[s^2-\min(x,y)],$$

$$\text{(3.2.6)}\quad C_1\big[(s-1)^2+(x-y)^2+(s-1)\big(1-\min(x,y)\big)\big] \leqslant Y^2 \leqslant C_2\big[(s-1)^2+(x-y)^2+(s-1)\big(1-\min(x,y)\big)\big],$$

$$\text{(3.2.7)}\quad 1 \leqslant s^2+\max(x,y) \leqslant Z_2 \leqslant C,$$

$$\text{(3.2.8)}\quad s^2-\min(x,y) \leqslant Z_1 \leqslant C[s^2-\min(x,y)].$$

In what follows we shall suppose that $1/2 < t < 1$ and $x \geqslant 0$; consequently, $1 < k < 2$. We are going to consider first the kernel A:

(3.2.9)
$$A = -(1+\alpha+\beta)k^{-2}t^{-(1+\alpha/\beta+\beta/2)} \int_0^{\pi/2} \frac{(1-t)(k\sec w)^{\alpha+\beta+2}}{Z_1^\alpha Z_2^\beta Y} \cos(\alpha-\beta)w\,dw.$$

By introducing the new variable $s = k\sec w$, we obtain (suppose $1/2 < t < 1$ and $x \geqslant 0$, note also that $1 < k < 2$)

$$|A| \leqslant (1+\alpha+\beta)k^{-1}\frac{t^{-1-\alpha/2-\beta/2}}{4}(1-t)\int_k^\infty \frac{s^{\alpha+\beta+1}}{Z_1^\alpha Z_2^\beta Y}\frac{ds}{\sqrt{s^2-k^2}} \tag{3.2.10}$$
$$\leqslant C(\alpha,\beta)+C(\alpha,\beta)(1-t)\int_k^2 \frac{1}{Z_1^\alpha Z_2^\beta Y}\frac{ds}{\sqrt{s-k}}.$$

In the same way we get for B, C, and D, respectively:

$$|B| \leqslant C(\alpha,\beta)+C(\alpha,\beta)(1-t)\int_k^2 \frac{2s^2-1-xy}{Z_1^\alpha Z_2^\beta Y^3}\frac{ds}{(s-k)^{1/2}}, \tag{3.2.11}$$

(3.2.12)
$$|C| \leqslant C(\alpha,\beta)+C(\alpha,\beta)(1-t)\int_k^2 \frac{1}{Z_1^{\alpha+1} Z_2^\beta Y}\left(1+\frac{2s^2-(1+xy)}{Y}\right)\frac{ds}{(s-k)^{1/2}},$$

(3.2.13)
$$|D| \leqslant C(\alpha,\beta)+C(\alpha,\beta)(1-t)\int_k^2 \frac{1}{Z_1^{\alpha} Z_2^{\beta+1} Y}\left(1+\frac{2s^2-(1+xy)}{Y}\right)\frac{ds}{(s-k)^{1/2}}.$$

But notice that $Y \leqslant Z_i \leqslant C$. Therefore our four terms are dominated by

$$E = C(\alpha,\beta)+C(\alpha,\beta)(1-t)\int_k^2 \frac{(s-\min(x,y))}{Z_1^\alpha Z_2^\beta Y^3}\frac{ds}{(s-k)^{1/2}}, \tag{3.2.14}$$

as it follows by formulas (3.2.1) to (3.2.8).

Our next task will be to estimate the integral

$$I = (1-t)\int_k^2 \frac{1}{[s-\min(x,y)]^\alpha} \times \tag{3.2.15}$$
$$\times \frac{1}{\left[(x-y)^2+(s-1)^2+(s-1)(1-\min(x,y))\right]^{3/2}}\,\frac{s-\min(x,y)}{(s-k)^{1/2}}\,ds.$$

In order to simplify the notation let us introduce φ to be defined as

$$\varphi(x,t) = (k-1)^{1/2}[(k-x)]^{1/2}. \tag{3.2.16}$$

LEMMA 2. *The following estimate for I is valid:*

$$I \leqslant C(a,\beta)\sum_{n=0}^{\infty}\frac{1}{2^{n/2}}\frac{1}{J^{a,\beta}\big(I_n(x,t)\big)}X_{I_n(x,t)}(y); \tag{3.2.17}$$

$X_{I_n(x,t)}$ *denotes the characteristic function of the interval $I_n(x,t)$ and $I_n(x,t)$ stands for the interval*

$$[x-2^n\varphi(x,t),\ x+2^n\varphi(x,t)]\cap[-1,1].$$

Proof. First assume that $|x-y|<\varphi(x,t)$. Then the following estimate is valid:

$$I \leqslant (1-t)\int_k^2 \frac{[s-\min(x,y)]}{[s-\min(x,y)]^a(s-1)^{3/2}[s-\min(x,y)]^{3/2}}\frac{ds}{(s-k)^{1/2}} \tag{3.2.18}$$

$$\leqslant (1-t)\int_k^2 \frac{1}{[k-\min(x,y)]^{a+1/2}(s-1)^{3/2}}\frac{ds}{(s-k)^{1/2}}.$$

An integration by parts yields the following inequality:

$$I \leqslant C(a)\frac{(1-t)}{[k-\min(x,y)]^{a+1/2}}\left\{\int_k^2\frac{(s-k)^{1/2}}{(s-1)^{5/2}}ds + C(a)\right\} \tag{3.2.19}$$

$$\leqslant C(a)\frac{(1-t)}{[k-\min(x,y)]^{a+1/2}}\frac{1}{(k-1)}.$$

Notice that for $1-x>\varphi(x,t)$ we have

$$J^{a,\beta}\{I_0(x,t)\} \leqslant C(a,\beta)\{[(1-x)+\varphi(x,t)]^{a+1}-[(1-x)-\varphi(x,t)]^{a+1}\} \tag{3.2.20}$$

and for $1-x\leqslant\varphi(x,t)$ we have

$$J^{a,\beta}\{I_0(x,t)\} \leqslant C(a,\beta)[(1-x)+\varphi(x,t)]^{a+1}. \tag{3.2.21}$$

In both cases the following inequality can be readily checked:

$$J^{a,\beta}\{I_0(x,t)\} \leqslant C(a,\beta)(k-x)^a\varphi(x,t). \tag{3.2.22}$$

By using (3.2.19) and (3.2.22) we get (notice that $(k-1)\sim(1-t)^2$)

$$I \leqslant \frac{C(a,\beta)}{J^{a,\beta}\{I_0(x,t)\}}X_{I_0(x,t)}(y). \tag{3.2.23}$$

We were dealing with the case $|x-y|<\varphi(x,t)$ and now we are going to deal with the general case:

$$n\in N,\quad 2^{n-1}\varphi(x,t)<|x-y|\leqslant 2^n\varphi(x,t).$$

We distinguish two situations:

$$(3.2.24)\qquad \begin{array}{ll} \text{(a)} & (1-x) < 2^{n-1}\varphi(x,t), \\ \text{(b)} & (1-x) \geqslant 2^{n-1}\varphi(x,t). \end{array}$$

Situation (a). Notice that $y < x$. Therefore, since $(k-1) < \varphi(x,t) < |x-y|$, it follows that

$$(3.2.25)\qquad I \leqslant C(a,\beta)\Bigg\{(1-t)\Bigg[\int_k^{1+|x-y|} \frac{1}{\big(s-\min(x,y)\big)^{a-1}|x-y|^3}\,\frac{ds}{(s-k)^{1/2}} + \int_{1+|x-y|}^{2} \frac{1}{\big(s-\min(x,y)\big)^{a+1/2}(s-1)^{3/2}}\,\frac{ds}{(s-k)^{1/2}}\Bigg]\Bigg\}$$

$$\leqslant C(a,\beta)\left\{\frac{(1-t)}{(k-y)^{a-1}}\,\frac{|x-y|^{1/2}}{|x-y|^3} + \frac{(1-t)}{(k-y)^{a+1/2}}\int_{1+|x-y|}^{2}\frac{ds}{(s-1)^{3/2}(s-k)^{1/2}}\right\}.$$

An integration by parts yields

$$(3.2.26)\qquad I \leqslant C(a,\beta)\left\{\frac{(1-t)}{(k-y)^{a-1}}\,\frac{1}{|x-y|^{5/2}} + \frac{(1-t)}{(k-y)^{a+1/2}}\left[C + \frac{1}{|x-y|} + \int_{1+|x-y|}^{2}\frac{1}{(s-1)^2}\,ds\right]\right\} \leqslant C(a,\beta)\left\{\frac{(1-t)}{|x-y|^{a+3/2}}\right\}.$$

We have used the fact that $k-y$ is of the order of magnitude of $x-y$.

In what we have done, we have obtained a convenient estimate for I. In what follows we shall find a suitable estimate for $J^{a,\beta}\{I_n(x,t)\}$.

Since $1 \in I_n(x,t)$, we have the inequality

$$(3.2.27)\qquad J^{a,\beta}\big(I_n(x,t)\big) \leqslant C(a,\beta)[(1-x)+2^n\varphi(x,t)]^{a+1} \leqslant C(a,\beta)2^{a+1}[2^n\varphi(x,t)]^{a+1} \leqslant C(a,\beta)|x-y|^{a+1},$$

where we used the fact that

$$J^{a,\beta}([a,1]) \leqslant C(a,\beta)\cdot(1-a)^{a+1}.$$

On the other hand, we have also

$$(3.2.28)\qquad \frac{(1-t)}{|x-y|^{1/2}} < \frac{\varphi(x,t)^{1/2}}{[2^{n-1}\varphi(x,t)]^{1/2}} = \frac{1}{2^{(n-1)/2}}.$$

By combining (3.2.26)-(3.2.28) we get for situation (a)

$$(3.2.29)\qquad I \leqslant C(a,\beta)\frac{1}{2^{n/2}}\,\frac{1}{J^{a,\beta}\{I_n(x,t)\}}\,\chi_{I^n(x,t)}(y).$$

Situation (b) will be splitted into two subcases (b_1) and (b_2):

$$(b_1) \qquad (k-1) < \frac{(x-y)^2}{(1-x)},$$

$$(b_2) \qquad (k-1) \geqslant \frac{(x-y)^2}{(1-x)}.$$

Subcase (b_1). We have

$$(3.2.30) \quad I \leqslant C(a,\beta)(1-t)\left\{ \int\limits_k^{1+(x-y)^2/(1-x)} \frac{s-\min(x,y)}{[s-\min(x,y)]^a[x-y]^3} \frac{ds}{(s-k)^{1/2}} + \right.$$
$$\left. + \int\limits_{1+(x-y)^2/(1-x)}^{2} \frac{s-\min(x,y)}{[s-\min(x,y)]^{a+3/2}(s-1)^{3/2}} \frac{ds}{(s-k)^{1/2}} \right\}$$
$$\leqslant C(a,\beta)(1-t)\left\{ \frac{1}{(1-x)^{a-1}(x-y)^3} \int\limits_k^{1+(x-y)^2/(1-x)} \frac{ds}{(s-k)^{1/2}} + \right.$$
$$\left. + \frac{1}{(1-x)^{a+1/2}} \int\limits_{1+(x-y)^2/(1-x)}^{2} \frac{ds}{(s-1)^{3/2}(s-k)^{1/2}} \right\}.$$

As before, an integration by parts yields

$$(3.2.31) \quad I \leqslant C(a,\beta) \frac{(1-t)}{(1-x)^{a-1/2}|x-y|^2} \leqslant C(a,\beta) \frac{\varphi(x,t)}{(1-x)^a|x-y|^2}.$$

As in the previous case, we shall give an estimate for $J^{a,\beta}\{I_n(x,t)\}$:

$$(3.2.32) \qquad J^{a,\beta}\{I_n(x,t)\} \leqslant C(a,\beta)(1-x)^a 2^n \varphi(x,t).$$

Recalling that $|x-y| > 2^{n-1}\varphi(x,t)$, we get

$$(3.2.33) \qquad I \leqslant \frac{C(a,\beta)}{2^n} \frac{1}{J^{a,\beta}(I_n(x,t))} \chi_{I_n(x,t)}(y).$$

Subcase (b_2). In this subcase

$$(3.2.34) \qquad I = (1-t)\int\limits_k^2 \leqslant (1-t) \int\limits_{1+(x-y)^2/(1-x)}^{2}.$$

This last integral was already evaluated in (3.2.30) and one can obtain the same type of estimate as in (3.2.33). This completes the proof of the lemma.

4. Proof of theorems 1.2 and 1.3. By using lemma 2, for the multiple Watson kernel we have

(4.1.1)

$$K^{a,\beta}(r, X, Y) \leqslant C(a, \beta) \sum_{n_1,\dots,n_k} \frac{1}{2^{n_1/2}\dots 2^{n_k/2}} \frac{1}{J^{a,\beta}\{I_{\vec{n}}(X, r)\}} X_{I_{\vec{n}}(X,r)}(Y),$$

where

(4.1.2) $$\vec{n} = (n_1, n_2, \dots, n_k),$$

$X_Q(Y)$ is the characteristic function of Q, and

$$I_{\vec{n}}(X, r) = I_{n_1}(x_1, r_1) \times I_{n_2}(x_2, r_2) \times \dots \times I_{n_k}(x_k, r_k).$$

We introduce the following collection of maximal functions:

(4.1.3) $$M_{\vec{n}}(f)(x) = \sup_t \frac{1}{J^{a,\beta}\{I_{\vec{n}}(x, r(t))\}} \int_{I_n(x,r(t))} |f(Y)|\, dJ^{a,\beta}.$$

An application of lemma 1 to the family of rectangles $\{I_{\vec{n}}(X, r(t))\}$ with

(4.1.4) $$\varphi_j(t) = k(r_j(t))-1, \qquad h_j(x_j) = 1-x_j, \qquad K_j = 2^{n_j},$$

yields the weak type estimate

(4.1.5) $$J^{a,\beta}\{M_{\vec{n}}(f)(x) > \lambda\} \leqslant \frac{C(\vec{n})}{\lambda} \|f\|_{1,(J^{a,\beta})},$$

where

$$C(\vec{n}) \leqslant C \prod_{j=1}^{k} n_j.$$

Consequently, for $p > 1$, we have

(4.1.6) $$\|M_{\vec{n}}(f)\|_{p(J^{a,\beta})} \leqslant C' \frac{1}{(p-1)} \prod_{j=1}^{k} n_j \|f\|_{p(J^{a,\beta})}.$$

Theorem 1.2 follows from lemma 1.3 of [4], p. 121, and from estimates (4.1.1), (4.1.5) and (4.1.6).

Theorem 1.3 follows from theorem 1.2 and from the fact that for a dense subset of $L^p(J^{a,\beta})$, $1 \leqslant p < \infty$, the operator converges everywhere. Indeed, the set of multiple Jacobi series having only a finite number of non--vanishing terms may be chosen as a dense set. This completes both proofs.

REFERENCES

[1] H. Bateman, *Higher transcendental functions III*, edited by A. Erdélyi et al., New York - Toronto - London 1955.
[2] L. A. Caffarelli and C. P. Calderón, *Weak type estimates for the Hardy-Littlewood maximal functions*, Studia Mathematica 49 (1974), p. 217-224.
[3] C. P. Calderón, *On Abel summability of multiple Laguerre series*, ibidem 33 (1969), p. 273-294.
[4] — *Some remarks on the multiple Weierstrass transform and Abel summability of multiple Fourier-Hermite series*, ibidem 32 (1969), p. 119-148.
[5] B. Muckenhoupt and E. M. Stein, *Classical expansions and their relation to conjugate harmonic functions*, Transactions of the American Mathematical Society 118 (1965), p. 17-92.
[6] G. Szegö, *Orthogonal polynomials*, American Mathematical Society Colloquium Publications 23 (1959).

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