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GENERALIZED MIDSET PROPERTIES CHARACTERIZE GEODESIC CIRCLES AND INTERVALS

BY

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1. Introduction. A set is called a *singleton* if it is a one-point set and it is called a *doubleton* if it consists of two points. Let λ be a positive real number, and let a and b be two points of a metric space (X, d). The *λ-set*, $\lambda(a, b)$, of a and b is the set $\{x \mid d(a, x) = \lambda d(b, x)\}$, and X is said to have the *double λ-set property* (DλP) if there is a number λ such that, whenever a and b lie in X, $\lambda(a, b)$ is a doubleton. If $\lambda = 1$, then the DλP becomes the *double midset property* (DMP). There is no generality lost in assuming that $0 < \lambda \leqslant 1$ since the sets $\lambda(a, b)$ and $(1/\lambda)(b, a)$ are identical.

A subset S of a metric space is said to be a *segment* if S is isometric to a closed interval in E^1, and a segment with endpoints a and b will be denoted by $S(a, b)$. A point r is called a *ramification point* of X if there exist three points p, q, and q' such that r is a midpoint of p and q, and r is also a midpoint of p and q'. A point r is *between* two points p and q means that

$$p \neq r \neq q \quad \text{and} \quad d(p, r) + d(r, q) = d(p, q),$$

and r is a *midpoint* of p and q if r is between p and q, and $d(p, r) = d(q, r)$. In a segmentally connected metric space there is always a segment $S(p, q)$ containing r whenever r is between p and q (see [5], Lemma 15.1, p. 44). A *segmentally connected metric space* is one in which each two points lie in some segment. Every convex complete metric space is segmentally connected (see [5], Theorem 14.1, p. 41) but not conversely. A segmentally connected metric space has a ramification point if and only if there exist distinct segments $S(p, q)$ and $S(p, q')$ having a common midpoint r. A set X is *non-degenerate* if it contains more than one point.

With these definitions we are now able to give more precision to the statements of our results. Let X be a segmentally connected metric space with no ramification points. Theorem 3.1 states that X is isometric to a geodesic circle if and only if some midset in X is a doubleton; Theorem 3.3 states that X is either isometric to a geodesic circle or to a connected subset of E^1 if and only if X contains a doubleton λ-set for some $\lambda \in (0, 1)$; and Theorem 3.4 states that no midset in X can be countable unless it is a singleton or doubleton.

In Section 4 we consider a segmentally connected metric space X having the 0-dimensional midset or λ-set property. A metric space X has the *0-dimensional λ-set property* (0-λP) if there is a $\lambda \in (0, 1]$ such that all λ-sets in X are 0-dimensional sets. The 0-λP reduces to the *0-dimensional midset property* (0-MP) of [7] if $\lambda = 1$. A space X has the *unique λ-set property* (UλP) if there is a $\lambda \in (0, 1]$ such that every λ-set in X is a singleton, and the special case where $\lambda = 1$ is called the *unique midset property* (UMP). An example is given showing that the 0-λP is not strong enough to rule out ramification points in X; hence it cannot be used to characterize geodesic circles and subsets of E^1. However, the 0-MP does yield characterizations analogous to those of Section 3. These characterizations are given in Theorem 4.1 and in Theorem 4.2 and its corollaries.

Berard [2] showed that a connected metric space is homeomorphic to a subset of E^1 if it has the UMP, and he later proved [3] that a convex complete metric space is a topological simple closed curve if it has the DMP. Loveland and Valentine [7], and later Berard and Nitka [4], extended this result by showing that a segmentally connected metric space is isometric to a geodesic circle if it has the DMP (a short proof of this result can be found in [8]). However, the isometry exists with the weaker hypothesis that X contain just one double midset as long as X has no ramification points (Theorem 3.1). A segmentally connected metric space X with the 0-MP is isometric to a geodesic circle if and only if some midset is a doubleton (Theorem 4.1), it is isometric to a connected subset of E^1 if and only if some midset is a singleton (Theorem 4.1), and it can have no countable midset unless it lies in E^1 or is a geodesic circle (Lemma 2.2 and Theorems 3.4 and 4.1). In addition to these nice characterizations of subsets of E^1 and circles we are able to prove Theorem 3 of [7] with the considerable less hypothesis (see Corollary 4.3); in fact, we prove an analogue where only one midset is required to be 0-dimensional (Theorem 4.2).

Buseman [6] characterized hyperbolic and Euclidean spaces among his G-spaces using "flat" bisectors ("bisectors" are the same as "midsets"), and Beem [1] has announced that a pseudo-Riemannian manifold M has a constant curvature if and only if midsets are totally geodesic submanifolds of M. Wilker [10] has studied the connectivity of the midsets of connected subsets of Euclidean spaces.

2. Basic facts concerning λ-sets. A nice tool for use when dealing with λ-sets is the separation of space by each λ-set. This standard separation is given in Lemma 2.1, and the proof is a straightforward application of the continuity of the metric function. Lemma 2.2 is the key to the relation between the 0-MP and the absence of ramification points.

LEMMA 2.1. *If a and b are two points of a connected metric space X and $\lambda \in (0, 1]$, then $X - \lambda(a, b)$ is the union of two disjoint open sets L and R, where*

$$L = \{x \mid d(a, x) < \lambda d(b, x)\} \quad \textit{and} \quad R = \{x \mid d(a, x) > \lambda d(b, x)\}.$$

LEMMA 2.2. *If X is a segmentally connected metric space with the* 0-MP, *then X has no ramification points.*

Proof. Suppose that r is a ramification point of X. By definition there exist three points p, q, and q' such that r is a midpoint of p and q, and also a midpoint of p and q'. Let S and S' be segments from p to q and from p to q', respectively, such that $r \in S \cap S'$. Since $q \neq q'$, the last point r' of $S \cap S'$ is an interior point of both S and S'. Thus two points a and b exist between r' and q' and between r' and q, respectively, such that $d(a, r') = d(b, r')$. If t is a point of S between p and r', then

$$d(b, t) = d(b, r') + d(r', t)$$

since $\{t, b\} \subset S$ and, similarly,

$$d(a, t) = d(a, r') + d(r', t)$$

since $\{a, t\} \subset S'$. Now it is clear that $d(b, t) = d(a, t)$, so $t \in M(a, b)$. This implies that a segment $S(p, r')$ lies in $M(a, b)$, contradicting the 0-MP.

LEMMA 2.3. *Let X be a segmentally connected metric space without ramification points, and let $\lambda \in (0, 1]$. If a and b are two points of X, then no segment with either a or b as an endpoint can contain two points of $\lambda(a, b)$.*

Proof. Suppose that there is a segment $S(a, c)$ containing two points m and n of $\lambda(a, b)$, where m is between a and n. It follows that

$$d(m, n) = d(a, n) - d(a, m) = \lambda\big(d(b, n) - d(b, m)\big) \leqslant \lambda d(m, n),$$

which implies $\lambda = 1$. But if $\lambda = 1$, it is easy to show (using $\{m, n\} \subset M(a, b)$) that a segment $S(b, n)$ containing m also exists, and then $S(b, n) \cup S(a, c)$ must contain a ramification point.

Now suppose that a segment $S(b, c)$ contains two points m and n of $\lambda(a, b)$, where m is between b and n. Then

$$d(m, n) = d(b, n) - d(b, m) = \frac{1}{\lambda}\big(d(a, n) - d(a, m)\big) \leqslant \frac{1}{\lambda} d(m, n),$$

so again $\lambda = 1$ and the argument above yields the same contradiction.

LEMMA 2.4. *Let X be a segmentally connected metric space with no ramification points, and let $\lambda \in (0, 1)$. If a and b are two points of X, then no segment containing b can contain two points of $\lambda(a, b)$.*

Proof. Suppose that m and n are two points of $\lambda(a, b)$ in a segment containing b. If b is between m and n, then

$$d(b, m)+d(b, n) = d(m, n)$$

from which we have

$$d(a, m)+d(a, n) = \lambda d(m, n) < d(m, n),$$

contrary to the triangle inequality. If b is not between m and n, then Lemma 2.3 applies.

3. The double midset and λ-set properties. The setting for this section is a segmentally connected metric space with no ramification points. We obtain characterizations of nice spaces using the DMP and the DλP. The main results are Theorems 3.1 and 3.3; Corollary 3.1 to Theorem 3.1 is the DMP characterization given by Loveland and Valentine (see [7], Theorem 2, and [8]) and by Berard and Nitka [4]. Corollary 3.3 to Theorem 3.3 is the analogue of Corollary 3.1 for λ-sets. Theorem 3.4, giving limits on the cardinality of a λ-set when X has no ramification points, is used along with Theorems 3.1, 3.2 and 3.3 in Section 4.

THEOREM 3.1. *Let X be a segmentally connected metric space with no ramification points. Then X is isometric to a geodesic circle if and only if some midset in X consists of two points.*

Proof. Since one direction of the proof is clear, we assume that X contains two points a and b such that $M(a, b) = \{m_1, m_2\}$. One of these points, say m_1, must lie in a segment S with endpoints a and b. Let S' be a segment with endpoints m_1 and m_2. If S' contains either a or b, then in Case 1 below it is shown that X is isometric to a geodesic circle. Case 2 shows that Case 1 always applies.

Case 1. Suppose that S' contains b (the argument is similar if S' contains a). Since

$$d(m_2, b)+d(b, m_1) = d(m_1, m_2) \quad \text{and} \quad \{m_1, m_2\} \subset M(a, b),$$

we see that a is also between m_1 and m_2. Let S'' be a segment with endpoints m_1 and m_2 such that $a \in S''$, let $X - M(a, b) = L \cup R$, where $a \in L$ and $b \in R$ as in Lemma 2.1, and let c and d be the midpoints of S' and S'', respectively. Since $c \in R$ and $d \in L$, a segment $W(c, d)$ must contain either m_1 or m_2, and both m_1 and m_2 lie between c and d. Thus distinct segments W and W' exist, with endpoints c and d, such that $m_1 \in W$ and $m_2 \in W'$. Let $C = W \cup W'$.

Suppose that there is a point $y \in X - C$. Then $y \notin M(a, b)$ and we may assume, for convenience in notation, that $y \in R$. Since $d \in L$, a segment T with endpoints d and y must contain either m_1 or m_2. But then $T \cup W \cup W' \cup S'$ would contain a ramification point. Thus we know that X is a convex simple closed curve C and it follows (see the proof of Theorem 2 of [7]) that X is isometric to a geodesic circle.

Case 2. Suppose that S' contains neither a nor b. We may assume that

$$d(a, t) > d(b, t) \quad \text{for all } t \in S' - \{m_1, m_2\},$$

since $S' \cap M(a, b) = \{m_1, m_2\}$. Let $\{t_i\}$ be a sequence of points of S' converging to m_1, and, for each i, let S_i be a segment from a to t_i. If each S_i contains m_2, then

$$d(a, m_2) + d(m_2, t_i) = d(a, t_i) \quad \text{for each } i.$$

The continuity of the metric would then imply the existence of a segment $S(a, m_1)$ containing m_2, contrary to Lemma 2.3. No S_i can contain m_1 unless it also contains b, for, otherwise, $S_i \cup S$ would contain a ramification point. However, if m_1 and b lie in S_i for each i, then

$$d(t_i, b) + d(b, m_1) = d(t_i, m_1) \quad \text{for each } i,$$

and this leads to the contradiction that $d(m_1, b) = 0$, since $t_i \to m_1$. Thus, for some i, S_i contains neither m_1 nor m_2. But this contradicts Lemma 2.1 since the endpoints of S_i are separated by $M(a, b)$.

COROLLARY 3.1 (see [8]). *A non-degenerate segmentally connected metric space X is isometric to a geodesic circle if and only if X has the* DMP

This is an immediate consequence of Theorem 3.1 since, by Lemma 2.2, the DMP will not permit ramification points.

THEOREM 3.2. *Let X be a non-degenerate segmentally connected metric space with no ramification points. Then X is isometric to a connected subset of E^1 if and only if X contains a singleton midset.*

Proof. Every midset in X is a singleton if $X \subset E^1$. In the other direction, we let a and b be two points of X such that $M(a, b) = \{m\}$, and let S be a segment with endpoints a and b. Suppose that there is a point y in X such that y lies in no segment containing both a and b. Since $y \neq m$, we may assume with no loss in generality that $d(a, y) > d(b, y)$. A segment T with endpoints a and y must intersect $M(a, b)$, so $m \in T$. However, $S \cup T$ now contains a ramification point. Thus an isometry taking X into E^1 is easily produced.

COROLLARY 3.2. *A non-degenerate segmentally connected metric space X is isometric to a connected subset of E^1 if and only if X has the* UMP.

Proof. The UMP rules out ramification points in X by Lemma 2.2, so this reduces immediately to Theorem 3.2.

THEOREM 3.3. *Let X be a non-degenerate segmentally connected metric space with no ramification points. Then X is isometric to a geodesic circle or to a connected subset of E^1 if and only if there is a real number $\lambda \in (0,1)$ such that X contains a doubleton λ-set.*

Proof. If X is isometric to a connected subset of E^1 and $\lambda \in (0,1)$, then points a and b in X are easily found such that $\lambda(a,b)$ is a doubleton. If X is isometric to a geodesic circle and $\lambda \in (0,1)$, then, for any two points a and b of X, $\lambda(a,b)$ will be a doubleton. Thus it suffices to show that these are the only segmentally connected metric spaces with no ramification points having a doubleton λ-set for some $\lambda \in (0,1)$.

Let a and b be two points of X such that $\lambda(a,b)$ is a doubleton $\{m_1, m_2\}$ for some $\lambda \in (0,1)$, where m_1 lies in a segment S with endpoints a and b. We shall show that the midset $M(a,b)$ contains at most two points. Theorems 3.1 and 3.2 can then be applied to see that X is isometric either to a geodesic circle or to a connected subset of E^1.

Suppose that $M(a,b)$ contains three points p, q, and r, where $p \in S$ and $d(a,q) \leqslant d(a,r)$. Notice that $S \cap \{q, r, m_2\} = \emptyset$. Let

$$X - \lambda(a,b) = L \cup R,$$

where $a \in L$ and $b \in R$ as in Lemma 2.1. Since $\{q,r\} \subset R$ and $a \in L$, a segment $S(a,q)$ must intersect $\lambda(a,b)$. But X has no ramification points, so $S(a,q)$ cannot contain m_1 unless it contains b. If $S(a,q)$ contains b, then $d(a,q) > d(b,q)$ contrary to $q \in M(a,b)$. Thus $S(a,q)$ contains m_2 and, for the same reasons, $S(a,r)$ must also contain m_2. Again, X has no ramification points, so $d(a,q) \leqslant d(a,r)$ implies $S(a,q) \subset S(a,r)$. However, this contradicts Lemma 2.3, and $M(a,b)$ contains at most two points. Thus the result follows.

COROLLARY 3.3. *Let X be a non-degenerate segmentally connected metric space. Then X is isometric to a geodesic circle or to E^1 if and only if there is a real number $\lambda \in (0,1)$ such that X has the* DλP.

Proof. Suppose that r is a ramification point of X. As in the proof of Lemma 2.2, we obtain segments $S(p,q)$ and $S'(p,q')$ and a point r' such that r' is the "last point" of $S \cap S'$ and r' lies in the interior of both S and S'. We let $a = r'$ and we choose b between p and a such that

$$\frac{\lambda}{1-\lambda} d(a,b) < \min[d(a,q), d(a,q')].$$

A point x must exist between b and a such that

$$d(a,x) = \frac{\lambda}{1+\lambda} d(a,b),.$$

and points y and z exist between a, q and a, q', respectively, such that

$$d(a, y) = d(a, z) = \frac{\lambda}{1-\lambda} d(a, b).$$

Then $\lambda(a, b)$ contains $\{x, y, z\}$, contradicting the DλP. Thus X has no ramification points, and Theorem 3.3 applies to show that X is isometric either to a geodesic circle or to a subset I of E^1.

Suppose that X is isometric to I, and I is bounded from below. Let g be the greatest lower bound for I, and choose a number b in I such that $b \neq g$. Choose a between $\lambda b + (1-\lambda)g$ and g. It follows that a is between g and b since $0 < \lambda < 1$. It is easy to check that the λ-set $\lambda(a, b)$, relative to all of E^1, is

$$\left\{a - \frac{\lambda}{1-\lambda}(b-a),\ a + \frac{\lambda}{1+\lambda}(b-a)\right\},$$

but

$$a - \frac{\lambda}{1-\lambda}(b-a) < g.$$

This means that $\lambda(a, b)$, relative to I, contains only one point. Since I and X are isometric, this contradicts the DλP, and similarly I has no upper bound. Thus $I = E^1$.

THEOREM 3.4. *If X is a segmentally connected metric space with no ramification points, and a and b are two points of X such that, for some $\lambda \in (0, 1]$, $\lambda(a, b)$ contains three points, then $\lambda(a, b)$ is uncountable.*

Proof. Let m, m_1, and m_2 be three points in $\lambda(a, b)$, where m_1 belongs to a segment with endpoints a and b. Since X has no ramification points, it follows from Lemma 2.3 that either a is not between m_1 and m_2 or a is not between m_1 and m. We assume the points named so that a is not between m_1 and m. Suppose that b is between m_1 and m. By Lemma 2.4, $\lambda = 1$, and it then follows (from $\{m, m_1\} \subset M(a, b)$) that a is between m_1 and m, contrary to the above labelling. Since neither a nor b is between m and m_1, a segment $T'(m_1, m)$ can contain neither a nor b.

It follows from the continuity of the metric and from Lemma 2.3 that a subsegment $S(x, m_1)$ of T' exists such that no segment $S(a, t)$ contains m if $t \in S(x, m_1)$. By Lemma 2.3, m cannot lie between b and m_1, so, similarly, there is a subsegment $S(y, m_1)$ of $S(x, m_1)$ such that no segment $S(b, t)$ contains m if $t \in S(y, m_1)$. Since X has no ramification points, for each $t \in S(y, m_1)$ different from m_1, m_1 is neither between a and t nor between b and t; hence

$$\{m_1, m\} \cap \big(S(a, t) \cup S(b, t)\big) = \emptyset$$

for every two segments $S(a, t)$ and $S(b, t)$.

Suppose that $\lambda(a, b)$ is countable. Since $\lambda(a, b)$ is closed, it is clear that $S(y, m_1)$ contains a segment T such that

$$T \cap \lambda(a, b) = \emptyset.$$

Let $X - \lambda(a, b) = L \cup R$, where $a \in L$ and $b \in R$ as in Lemma 2.1. Since the cases are similar, we may assume that $T \subset R$. For a point $t \in T$ and a segment $S(a, t)$ containing p, put

$$D = \{p \in \lambda(a, b) \mid t \in T,\ p \in S(a, b)\},$$

and define $f: D \to T$ by letting $f(p)$ be a point t such that $p \in S(a, t)$. To see that f is well defined we suppose that there exist distinct segments $S(a, t)$ and $S(a, t')$ both containing a point p of $\lambda(a, b)$. By the construction of T, neither segment intersects $\{m, m_1\}$, so $T' \cup S(a, t) \cup S(a, t')$ must contain a ramification point. Thus only one segment $S(a, t)$ can contain p. Since X is segmentally connected, it is clear that, for each $t \in T$, X contains a segment $S(a, t)$ and, since $\lambda(a, b)$ separates a from t, $S(a, t)$ contains a point p of D. Then $f(p) = t$ and f is surjective. Since T is uncountable and f is surjective, D must be uncountable. Of course, this makes $\lambda(a, b)$ uncountable and this contradiction completes the proof.

4. The 0-dimensional midset property. In the preceding section we showed how the more general DλP could replace the DMP, previously considered in [3] and [7], to yield characterizations of a circle or the line E^1. This section is devoted to an analysis of spaces in which, for some $\lambda \in (0, 1]$, some or all of the λ-sets are 0-dimensional. First we give an example to show that the 0-dimensional λ-set property for $\lambda \neq 1$ is not enough to rule out ramification points, and hence one cannot expect the 0-λP to yield only circles or subsets of E^1.

Example 4.1. Let X be the union of three rays with a common initial point in a Euclidean space. Define the distance between two points to be their Euclidean distance if they lie in the same ray and to be the length of the shortest path between them otherwise. Then X is a complete, convex, locally externally convex metric space with the 0-λP if $\lambda < 1$. Furthermore, if $\lambda < 1$, then λ-sets are either doubletons or three-point sets. The example, of course, can be generalized by starting with more rays.

Berard [2] proved that a connected metric space is homeomorphic to a subset of E^1 if it has the UMP. An attempt to use the UλP in a geometric characterization of subsets of a line is futile since no space containing a segment has the UλP unless $\lambda = 1$. In view of this, Example 4.1, and our desire to obtain spaces without ramification points, it is clear that we should restrict λ to 1 and consider only the 0-MP. Our first result is a quick consequence of Lemma 2.2 and Theorems 3.1 and 3.2. Theorem 4.1, part (1), can be viewed as a generalization of the main result of [8].

THEOREM 4.1. *Let X be a segmentally connected metric space with the* 0-MP. *Then*

(1) *X is isometric to a geodesic circle if and only if some midset is a doubleton;*

(2) *X is non-degenerate and isometric to a subset of E^1 if and only if some midset is a singleton.*

In [7] a rather complicated "0-dimensional weak linear midset property (0-WLMP)" was defined and it was proved (Theorem 3 of [7]) that a non-degenerate complete, convex, locally externally convex metric space with the 0-WLMP is isometric either to a geodesic circle or to E^1. We obtain the same conclusion using the 0-MP in place of the 0-WLMP (Corollary 4.3). In fact, only one midset must be 0-dimensional if X is known to have no ramification points (Theorem 4.2). First we give some definitions and lemmas.

A metric space X is said to be *locally externally convex* if for each point p of X there is a neighborhood N of p such that whenever x and y are two points of N there is a point z such that y is between x and z. A segment is *maximal* if it is a proper subset of no segment. We use $B(x, \xi)$ to denote $\{y \mid d(x, y) < \xi\}$. The proofs of the next two lemmas are not difficult.

LEMMA 4.1. *If m_1 is a point of a segmentally connected, locally externally convex metric space X, then there is a positive number ξ such that no segment in $B(m_1, \xi)$ is maximal.*

LEMMA 4.2. *If a segmentally connected metric space has no ramification points, then non-maximal segments are unique.*

THEOREM 4.2. *Let X be a segmentally connected, locally externally convex metric space without ramification points. If there exist two points a and b of X such that $M(a, b)$ is 0-dimensional, then X is isometric either to a geodesic circle or to a connected open subset of E^1.*

Proof. We show that $M(a, b)$ can contain no more than two points and then appeal to Theorems 3.1 and 3.2 to show that X is either isometric to a geodesic circle or to a connected subset of E^1. However, a connected non-degenerate subset of E^1 that is locally externally convex must be open.

Suppose that $M(a, b)$ contains three points m_1, m_2, and m_3, where m_1 belongs to a segment S joining a and b. We use Lemma 2.3, just as in the proof of Theorem 3.4 (with $\lambda = 1$), to obtain a segment $T'(m, m_1)$ containing neither a nor b, where $m \in \{m_2, m_3\}$. Consequently, $T' \cap S = \{m_1\}$. By Lemma 4.1, there is a $\xi > 0$ small enough that

$$\{a, b, m\} \cap B(m_1, \xi) = \emptyset$$

and no segment in $B(m_1, \xi)$ is maximal. Let $X - M(a, b) = L \cup R$, where $a \in L$ and $b \in R$. Choose points x and y in $S \cap B(m_1, \xi/2)$ such that $x \in L$ and $y \in R$. A subsegment T of $T'(m_1, m)$ exists such that

$$T \cap M(a, b) = \emptyset \quad \text{and} \quad T \subset B(m_1, \xi/2).$$

No generality is lost in assuming that $T \subset R$. For $S(x, t)$, put

$$B = \{p \in M(a, b) \mid p \in S(x, t),\ t \in T\},$$

and define $g\colon B \to T$ by letting $g(p)$ be the unique (by Lemma 4.2) point t of the segment $S(x, t)$, where $p \in S(x, t)$. To show that g is continuous, consider a sequence $\{p_i\}$ of points in B converging to p. By taking subsequences, if necessary, we may assume that $\{g(p_i)\}$ converges to a point $t \in T$. Suppose that $g(p) = t_0 \neq t$. Then

$$d(x, p_i) + d\big(p_i, g(p_i)\big) = d\big(x, g(p_i)\big) \quad \text{for each } i,$$

so by the continuity of the metric d we have

$$d(x, p) + d(p, t) = d(x, t).$$

However, this yields a ramification point since p also lies between x and t_0. It is clear now that g is a continuous surjection between a 0-dimensional set B and the 1-dimensional set T. However, this is a contradiction since g is also injective by the selection of $B(m_1, \xi)$ and Lemma 4.2.

COROLLARY 4.1. *If X is a non-degenerate complete, convex, locally externally convex metric space with no ramification points and if some midset in X is* 0*-dimensional, then X is isometric either to E^1 or to a geodesic circle.*

COROLLARY 4.2. *If X is a non-degenerate, segmentally connected, locally externally convex metric space with the* 0-MP, *then X is isometric either to a geodesic circle, to an open ray in E^1, to an open interval in E^1, or to E^1.*

Proof. Lemma 2.2 shows that X has no ramification points, and Theorem 4.2 then applies to eliminate X being any space except geodesic circles and subsets of E^1. The locally externally convex hypothesis eliminates all subsets of E^1 except those in the conclusion of Corollary 4.2.

COROLLARY 4.3. *If X is a non-degenerate complete, convex, locally externally convex metric space with the* 0-MP, *then X is isometric either to a geodesic circle or to E^1.*

QUESTION. Does Theorem 4.2 remain true with the words "locally externally convex" and "open" removed? (**P 1022**)

A metric space is said to have the *finite midset property* (FMP) if each of its midsets is a finite set, and X is said to have the *triple midset property* (TMP) if each midset in X consists of three points. Loveland and Wayment [9] have asked if there can be a non-degenerate connected metric space with the TMP. It follows from Lemma 2.2 and Theorem 3.4 that no non-degenerate segmentally connected metric space has the FMP (unless all midsets are singletons or all are doubletons); hence no such space can have the TMP.

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