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ON COVERINGS OF A UNIFORMITY

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In Isbell's book [1], p. 52, there is raised a question whether or not the coverings of a given uniformity, the cardinalities of which are less than a given cardinal number m, form a base for a uniformity. Shirota [3] was the first who answered positively to this question in the case of a fine uniformity for the given topology. The answer is known to be positive for $m = \aleph_0$ (cf. [1], exercise, p. 52); the answer is also positive for uniformities having bases consisting of point-finite coverings (cf. [1], Theorem 28, p. 69, or, in a more precise form, the paper [2] of Kulpa) and for uniformities having bases consisting of σ-point-finite coverings (cf. Vidossich [4]).

In this note we shall give a positive solution of Isbell's question, assuming the generalized continuum hypothesis (i.e., if $n < m$, then $2^n \leqslant m$ for all cardinal numbers).

A covering $\mathscr{V}$ is a *point-star-refinement* of a covering $\mathscr{U}$ if, for each x from X, there exists a U in $\mathscr{U}$ such that $\mathrm{st}(x, \mathscr{V}) \subset U$; it is a *star-refinement* of $\mathscr{U}$ if, for each V in $\mathscr{V}$, there exists a U in $\mathscr{U}$ such that $\mathrm{st}(V, \mathscr{V}) \subset U$; recall that a point-star-refinement, if applied two times, gives a star-refinement.

THEOREM. *Let μ be a uniformity on X and let m be an infinite cardinal number. Then the family of coverings from μ, the cardinalities of which are less than m, is a base for a uniformity ν, $\nu \subset \mu$.*

Proof. It suffices to show that, for each $\mathscr{U}$ from μ, there exists in μ a covering $\mathscr{W}$ being a point-star-refinement of $\mathscr{U}$ and such that $\mathrm{card}\,\mathscr{W} \leqslant \mathrm{card}\,\mathscr{U}$ or $\mathscr{W}$ is finite if $\mathscr{U}$ is finite.

Let λ be an initial ordinal number for the cardinality of $\mathscr{U}$. We can assume that $\mathscr{U} = \{U_\alpha : \alpha < \lambda\}$. Let $\mathscr{V}$ be an arbitrary star-refinement of $\mathscr{U}$ belonging to μ. Define $p(V)$ for $V \in \mathscr{V}$ to be the least α such that $U_\alpha \supset \mathrm{st}(V, \mathscr{V})$, and let $\mathscr{V}_\alpha = \{V : p(V) = \alpha\}$. We have

$$\mathrm{st}(V, \mathscr{V}) \subset U_\alpha \quad \text{for } V \in \mathscr{V}_\alpha. \tag{1}$$

For each α we define a partition of the collection $\mathscr{V}_\alpha$ as follows: elements V and V' of $\mathscr{V}_\alpha$ are in the same element of the partition iff

$$(2) \qquad V \subset U_\gamma \Leftrightarrow V' \subset U_\gamma \quad \text{for each } \gamma, \text{ where } \gamma \leqslant \alpha.$$

The set of all elements of the partition of $\mathscr{V}_\alpha$ is of the cardinality not greater than the cardinality of the family of all subsets of the set $\{\gamma: \gamma \leqslant \alpha\}$, and so, in virtue of the generalized continuum hypothesis, it is not greater than card $\mathscr{U}$ or is finite if $\mathscr{U}$ is finite.

Let $\mathscr{W}_\alpha$ be the collection consisting of unions of elements of the partition of $\mathscr{V}_\alpha$. Let $\mathscr{W} = \bigcup \{\mathscr{W}_\alpha: \alpha < \lambda\}$.

It is obvious that $\mathscr{W}$ is a covering belonging to μ (since $\mathscr{V}$ is a refinement of $\mathscr{W}$ and $\mathscr{V} \in \mu$) and that card $\mathscr{W} \leqslant$ card $\mathscr{U}$ or $\mathscr{W}$ is finite.

It remains to prove that $\mathscr{W}$ is a point-star-refinement of $\mathscr{U}$. To do this take $x \in X$. Let $\alpha(x) = \min\{\alpha: x \in W \in \mathscr{W}_\alpha\}$. We shall show that $\operatorname{st}(x, \mathscr{W}) \subset U_{\alpha(x)}$. From the definition of $\alpha(x)$ it follows that there exists a $V \in \mathscr{V}_{\alpha(x)}$ such that $x \in V$. By (1), we get $x \in V \subset \operatorname{st}(V, \mathscr{V}) \subset U_{\alpha(x)}$.

Let W be such that $x \in W \in \mathscr{W}_\alpha$. There exists V' in $\mathscr{V}_\alpha$ such that $x \in V'$ and such that V' belongs to that element of the partition of $\mathscr{V}_\alpha$ the union of which is W.

But $x \in V \cap V'$. Hence $V' \subset \operatorname{st}(V, \mathscr{V}) \subset U_{\alpha(x)}$. By definition (2) of the partition and the inequality $\alpha(x) \leqslant \alpha$, we infer that W is contained in $U_{\alpha(x)}$.

Since W is an arbitrary element of $\mathscr{W}$ such that $x \in W$, we infer that $\operatorname{st}(x, \mathscr{W}) \subset U_{\alpha(x)}$.

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