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# *INVARIANT SUBSETS OF NON-SYNTHESIS LEPTIN ALGEBRAS AND NON-SYMMETRY*

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**1. Leptin algebras. Formulation of the problem.** Let $A$ be a Banach algebra and let $G$ be a locally compact group acting strongly continuously on $A$ as a group of isometric isomorphisms:

$$A \ni a \to a^x \in A, \quad x \in G.$$

By a *Leptin algebra* we mean what is also known under an awkward name of the generalized $L^1$-algebra (1), the algebra $L^1(G, A)$ of Bochner integrable (with respect to the left invariant Haar measure) $A$-valued functions on $G$. The convolution on $L^1(G, A)$ is defined by

$$f * g(x) = \int f(xy)^{y^{-1}} g(y^{-1})\, dy. \tag{1.1}$$

Actually, in general, the convolution depends on a factor-system (cf. [5]), which in our constructions, however, will be trivial.

If $A$ has also an isometric involution $a \to a^*$ and $(a^x)^* = (a^*)^x$, then $L^1(G, A)$ is a Banach $*$-algebra with

$$f^*(x) = \Delta(x^{-1})\big(f(x^{-1})^x\big)^*, \tag{1.2}$$

where $\Delta$ is the modular function on $G$.

Leptin algebras proved to be a most useful tool in studying the $L^1$-algebras of locally compact groups $\Gamma$ which are extensions of $N$ by $G$, since if $\Gamma = GN$, then

$$L^1(\Gamma) = L^1\big(G, L^1(N)\big), \tag{1.3}$$

and if $N$ is only normal in $\Gamma$ and $G = \Gamma/N$, then (1.3) still holds but (1.1) must be modified by a factor-system (cf. [7] and [8] and the bibliography therein).

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(1) The name *Leptin algebra* was first used by Leinert [4].

Here again a question posed by Leptin, which has arisen in his study of symmetry of $L^1$ of group extensions [7], formulated in terms of Leptin algebras, finds its answer in this language and produces, as a by-product, an example of a second solvable locally finite group $G$ such that $L^1(G)$ is non-symmetric and the spectral radius

$$\nu(f) = \lim_{n\to\infty} \|f^n\|^{1/n}, \qquad f \in L^1(G),$$

is discontinuous on Hermitian elements. An example of a locally finite group, whose $L^1$-algebra is non-symmetric, has recently been produced by Fountain et al. [1], but their group is far from being solvable.

A Banach algebra $A$ is called ***radical*** if, for every $a$ in $A$,

$$\lim_{n\to\infty} \|a^n\|^{1/n} = 0,$$

and $A$ is *nilpotent of degree less than or equal to c* if

$$a_1 \ldots a_c = 0 \qquad \text{for all } a_1, \ldots, a_c \text{ in } A.$$

If a Banach $^*$-algebra is non-radical, then it is easy to see that there is also a Hermitian element $a$ in $A$ such that

$$\lim_{n\to\infty} \|a^n\|^{1/n} > 0.$$

If such an algebra is also symmetric, then it has at least one non-zero $^*$-representation (into the algebra of bounded operators on a Hilbert space); cf., e.g., [10].

It is easy to see that if $A$ is nilpotent of degree less than or equal to $c$, then so is $L^1(G, A)$, since by (1.1) we have

$$(1.4) \qquad f_1 * \ldots * f_n(x)$$
$$= \int \ldots \int f_1(xx_{n-1} \ldots x_1)^{(x_{n-1}\ldots x_1)^{-1}} f(x_1)^{(x_{n-1}\ldots x_2)^{-1}} \ldots f_n(x_{n-1})\, dx_1 \ldots dx_{n-1}$$
$$\text{for all } f_1, \ldots, f_n \text{ in } L^1(G, A).$$

In [7] Leptin formulated the question whether $L^1(G, A)$ must be radical if $A$ is radical. As a matter of fact he was interested in the following situation.

Let $H$ be a locally compact Abelian group and let $G$ act on $H$ as a group of automorphisms: $h \to h^x$, $x \in G$, $h \in H$. Suppose that $E$ is a closed subset of $H$ such that $E = E^x$ for all $x$ in $G$ and $E$ is a set of non-synthesis for $A(H)$. Then, clearly, the set $E$, its kernel $k(E) = \{a \in A(H)\colon a(E) = 0\}$ and the minimal ideal $j(E) = \mathrm{cl}\{a \in A(H)\colon a(U) = 0 \text{ for an open } U \supset E\}$ are all stable under the action of $G$ on $A(H)$. Let $A = k(E)/j(E)$. $G$ acts on $A$ as a group of isometric $^*$-automorphisms and we form the Leptin algebra $L^1(G, A)$. We then have

$$L^1(G, A) = L^1(G, k(E))/L^1(G, j(E))$$

(cf. [5]), and so $L^1(G, A)$ is a subquotient of

$$L^1(G, A(H)) = L^1(G, L^1(H)) = L^1(GH).$$

QUESTION (Leptin). Is $L^1(G, k(E)/j(E))$ radical?

In some particular cases, where $G = O_n$, $H = \boldsymbol{R}^{n+1}$ and $E = S^n$, the answer is positive, since $k(E)/j(E)$ is nilpotent of degree $\frac{1}{2}(n-1)n$ (cf. [11] and [7]).

In Section 3 we show that the answer is negative already when $G$ is a direct sum of countably many copies of the cyclic group of order 2 and $H$ is the Cantor group; then there exists a compact subset $E$ of $H$ such that $L^1(G, k(E)/j(E))$ is non-radical.

In Section 4 we present a remark of Horst Leptin which shows how this result can be used to prove non-symmetry of certain group algebras. We show also the discontinuity of the spectral radius on Hermitian elements of $L^1(G\hat{H})$, which proves that $L^1(G\hat{H})$ is non-symmetric even though $G\hat{H}$ is locally finite and second solvable.

**2. Malliavin's sets of non-synthesis.** Let $H$ be a compact Abelian group and let $A(H) = \{\varphi: \hat{\varphi} \in l^1(\hat{H})\}$ be its Fourier algebra. $A(H)$ is naturally embedded in its dual $PM(H)$: if $\psi \in A(H)$, we write

$$\langle \varphi, \psi \rangle = \int \varphi\psi.$$

A classical Malliavin result (cf. [9]) states that for every infinite $H$ there exists a real-valued function $\varphi$ in $A(H)$ such that, for every natural number $n$,

$$\|e^{iu\varphi}\|_{PM} = o(|u|^{-n}) \quad \text{as } |u| \to \infty.$$

If for such a $\varphi$ (necessarily non-zero) a $T_\varphi$ in $PM(H)$ is defined by

$$T_\varphi = \int_{\boldsymbol{R}} iue^{iu\varphi}du,$$

then, by an easy integration by parts, we have

$$\langle \varphi, T_\varphi \rangle = -\int \langle 1, e^{iu\varphi} \rangle du \tag{2.1}$$

and

(2.2)

$$\langle \psi\varphi^2, T_\varphi \rangle = \int \langle \psi\varphi^2, iue^{iu\varphi} \rangle du = -\int \langle \psi, \varphi e^{iu\varphi} \rangle du = \langle \psi, e^{iu\varphi} \rangle \Big|_{-\infty}^{+\infty} = 0.$$

For our purpose we modify such a $\varphi$ by selecting a $\lambda_0$ such that if

$$\varphi_0 = \varphi + \lambda_0,$$

then

$$T_{\varphi_0} = \int_{R} iu \exp[iu\varphi_0] du$$

satisfies

$$\langle \varphi_0, T_{\varphi_0} \rangle \neq 0, \tag{2.3}$$

$$\langle 1, T_{\varphi_0} \rangle = 0, \tag{2.4}$$

$$\langle \psi\varphi_0^2, T_{\varphi_0} \rangle = 0 \quad \text{for all } \psi \text{ in } A(H). \tag{2.5}$$

Since (2.5) follows only from the definition of $T_{\varphi_0}$, as in (2.2), for any $\lambda_0$, our $\lambda_0$ should be selected in such a way that only (2.3) and (2.4) are satisfied. This can be done as follows. We put

$$M(u) = \langle 1, e^{iu\varphi} \rangle.$$

Since $\varphi$ is real, $M(-u) = \overline{M(u)}$, whence

$$\hat{M}(\lambda) = \int_{R} \langle 1, e^{iu\varphi} \rangle e^{iu\lambda} du = \int_{R} \langle 1, e^{iu(\varphi+\lambda)} \rangle du$$

is a real-valued function vanishing at infinity. Thus for an extremum point $\lambda_0$ we have

$$\hat{M}(\lambda_0) \neq 0 \quad \text{and} \quad \frac{d}{d\lambda} \hat{M}(\lambda_0) = 0.$$

But, as in (2.1), $\hat{M}(\lambda_0) \neq 0$ is simply (2.3) and

$$\langle 1, T_{\varphi_0} \rangle = \int_{R} iu \langle 1, \exp[iu\varphi_0] \rangle du = \int_{R} iu M(u) \exp[iu\lambda_0] du$$
$$= \frac{d}{d\lambda} \hat{M}(\lambda_0) = 0.$$

It follows immediately from (2.3) and (2.5) that the ideal $I_0$ generated by $\varphi_0$ in $A(H)$ is different from the (closed) ideal $I$ generated by $\varphi_0^2$. Consequently, by Wiener's theorem, there is a point $x_0$ in $H$ such that $\varphi_0(x_0) = 0$. Multiplying $T_{\varphi_0}$ by a suitable constant, we get such a $T_0$ that the following equalities are satisfied:

$$\begin{gathered} \langle \varphi_0, T_0 \rangle = 1, \quad \langle \varphi_0, \delta_{x_0} \rangle = 0, \quad \langle 1, T_0 \rangle = 0, \\ \langle \psi\varphi_0^2, T_0 \rangle = 0 \quad \text{for all } \psi \in A(H). \end{gathered} \tag{2.6}$$

Now, if $E = \{x \in H: \varphi_0(x) = 0\}$, then $x_0 \in E$ and $T_0$ does not annihilate

$$k(E) = \{\psi \in A(H): \psi(E) = 0\}$$

but annihilates the ideal generated by $\varphi_0^2$ and, consequently, since $A(H)$ is regular, $T_0$ annihilates

$$j(E) = \text{cl}\{\psi \in A(H): \psi(U) = 0 \text{ for an open } U \supset E\}.$$

**3. Construction of the example.** Let $D$ be the direct sum of countably, infinitely many cyclic groups of order 2 with generators $a_1, a_2, \ldots$, respectively. Then every element $d$ in $D$ is uniquely defined by a finite subset $\boldsymbol{i}$ of natural numbers as follows:

$$d = a_{\boldsymbol{i}} = \sum_{i \in \boldsymbol{i}} a_i.$$

It so happens that the function $\varphi_0$, described in the previous section and satisfying (2.6), is somewhat easier to be constructed on $\hat{D}$, the compact dual of $D$ — the Cantor group. If for $x = (\varepsilon_1, \varepsilon_2, \ldots) \in \hat{D}$, $\varepsilon_j = 0, 1$, we write

$$a_i(x) = (-1)^{\varepsilon_i},$$

then a sequence $c_i$ can be selected so that

$$\sum_i |c_i| < \infty \quad \text{and} \quad \varphi_0(x) = \sum_i c_i a_{2i}(x) a_{2i+1}(x)$$

(cf. [3]).

Let, as before, $E = \varphi_0^{-1}(0)$.

We define $\boldsymbol{D}$ as the Cartesian product of groups $\hat{D}$ indexed by the elements of the group $D$:

$$\boldsymbol{D} = \mathop{\times}_{d \in D} H_d, \qquad H_d = \hat{D}.$$

Let also

$$\boldsymbol{E} = \mathop{\times}_{d \in D} E_d, \qquad E_d = E.$$

For every $d$ in $D$ we define a function $f_d$ in $A(\boldsymbol{D})$ by

$$f_d(\boldsymbol{x}) = \varphi_0(x_d), \qquad \text{where } \boldsymbol{x} = (x_d)_{d \in D} \in \boldsymbol{D}.$$

Clearly,

$$\boldsymbol{E} = \bigcap_{d \in D} f_d^{-1}(0).$$

We also note that $A(\boldsymbol{D})$ is the projective tensor product of $A(H_d)$:

$$A(\boldsymbol{D}) = \bigotimes_{d \in D} A(H_d).$$

For a finite subset $\boldsymbol{d}$ of $D$ we define a functional $F_{\boldsymbol{d}} \in PM(\boldsymbol{D})$ as

$$F_{\boldsymbol{d}} = \bigotimes_{d \notin \boldsymbol{d}} \delta_d \otimes \bigotimes_{d \in \boldsymbol{d}} F_d,$$

where $\delta_d = \delta_{x_0}$ and $F_d = T_0$ (both $x_0$ and $T_0$ as in (2.6)). In other words, if

$$f(\boldsymbol{x}) = \sum_{\chi \in \hat{\boldsymbol{D}}} c_\chi \chi(\boldsymbol{x}), \qquad \sum_\chi |c_\chi| < \infty,$$

where

$$\chi = \chi_{\boldsymbol{c}} = \prod_{c \in \boldsymbol{c}} \chi_c \in \hat{\boldsymbol{D}} \qquad \text{and} \qquad \langle \chi, T_0 \rangle = m_\chi, \qquad \chi \in D,$$

then for a finite subset $\boldsymbol{d}$ of $D$ we have

$$\langle f, F_{\boldsymbol{d}} \rangle = \sum_{\chi \in \hat{\boldsymbol{D}}} c_\chi \prod_{c \in \boldsymbol{c} \setminus \boldsymbol{d}} \chi_c(x_0) \prod_{c \in \boldsymbol{c} \cap \boldsymbol{d}} m_{\chi_d}.$$

Clearly,

$$\|F_{\boldsymbol{d}}\|_{PM(\boldsymbol{D})} = \|T_0\|^{|\boldsymbol{d}|}_{PM(\hat{D})}. \tag{3.1}$$

Now, for a finite subset $\boldsymbol{d}$ of $\boldsymbol{D}$ and a multiindex function $a\colon \boldsymbol{D} \to \boldsymbol{Z}^+$ with $\operatorname{supp} a = \boldsymbol{d}$ we write

$$f^a_{\boldsymbol{d}} = \prod_{d \in \boldsymbol{d}} f_d^{a(d)}.$$

By virtue of (2.6), a simple verification shows that

$$\langle f^a_{\boldsymbol{c}}, F_{\boldsymbol{d}} \rangle = \begin{cases} 1 & \text{if } \boldsymbol{c} = a^{-1}(1) = \boldsymbol{d}, \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

and, moreover,

$$\langle f f^2_{\boldsymbol{c}}, F_{\boldsymbol{d}} \rangle = 0 \quad \text{for all } f \in \boldsymbol{A}(\boldsymbol{D}),\ \boldsymbol{c}, \boldsymbol{d} \subset D.$$

This shows that the ideal generated by $f^2_{\boldsymbol{d}}$, $\boldsymbol{d} \subset D$, is annihilated by all $F_{\boldsymbol{d}}$, $\boldsymbol{d} \subset D$, and, consequently,

$$\langle j(\boldsymbol{E}), F_{\boldsymbol{d}} \rangle = 0 \quad \text{for all } \boldsymbol{d} \subset D. \tag{3.3}$$

Let $\boldsymbol{B} = k(\boldsymbol{E})/j(\boldsymbol{E})$. Of course, by (3.3), $F_{\boldsymbol{d}}$ are functionals on $\boldsymbol{B}$ with the same norm:

$$\|F_{\boldsymbol{d}}\|_{\boldsymbol{B}^*} = \|T_0\|^{|\boldsymbol{d}|}_{PM}.$$

Now we define the action of $D$ on $\boldsymbol{D}$ by

$$(x_c)^d_{c \in D} = (x_{c+d})_{c \in D}.$$

This defines an action of $D$ on $\boldsymbol{A}(\boldsymbol{D})$ and, since $\boldsymbol{E}^d = \boldsymbol{E}$ for all $d \in D$, $k(\boldsymbol{E})$ and $j(\boldsymbol{E})$ are stable under $D$, which, in consequence, gives rise to an action of $D$ on $\boldsymbol{B}$ by isometric $^*$-automorphisms, $f \to f^{(d)}$, $d \in D$, and we note that

$$f_c^{(d)} = f_{c-d}, \quad c, d \in D, \tag{3.4}$$

where for a function $f$ in $k(\boldsymbol{E})$ we denote its image in $k(\boldsymbol{E})/j(\boldsymbol{E})$ by $f$ again.

Consider the Leptin algebra $L^1(D, \boldsymbol{B})$.

THEOREM. *$L^1(D, \boldsymbol{B})$ is a non-radical Banach $^*$-algebra.*

Proof. Let $\Phi$ in $L^1(D, \boldsymbol{B})$ be defined by

$$\Phi(d) = \begin{cases} 2^{-k} f_0 & \text{if } d = a_k,\ k = 0, 1, 2, \ldots,\ a_0 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $n_k = 2^k - 1$, $k = 1, 2, \ldots$ We define sequences $d_1^k, \ldots, d_{n_k}^k$, where $k = 1, 2, \ldots$, by induction as follows:

$$d_1^1 = a_1,$$
$$d_1^k, \ldots, d_{n_k}^k = d_1^{k-1}, \ldots, d_{n_{k-1}}^{k-1}, a_k, d_1^{k-1}, \ldots, d_{n_{k-1}}^{k-1}.$$

It is obvious that for a fixed $k$ all $d_1^k, \ldots, d_{n_k}^k$ belong to the set $a_1, \ldots, a_k$ and that $a_j$ appears $2^{k-j}$ times in the sequence $d_1^k, \ldots, d_{n_k}^k$. It is also easy to verify that all $n_k$ elements

$$d_1^k + \ldots + d_{n_k}^k,\ d_2^k + \ldots + d_{n_k}^k,\ \ldots,\ d_{n_k}^k \quad \text{in } D$$

are different. In fact, first we note that this is equivalent to the fact that, for every $i$, $j$, $1 \leqslant i \leqslant j \leqslant n_k$,

$$d_i^k + d_{i+1}^k + \ldots + d_j^k \neq 0. \tag{3.5}$$

To see this we argue by induction on $k$:

if $i \leqslant n_{k-1} \leqslant j$, then $a_{k-1}$ appears only once in $d_i^k, \ldots, d_j^k$ and (3.5) follows;

if $i \leqslant j < n_{k-1}$, then $d_i^k + \ldots + d_j^k = d_i^{k-1} + \ldots + d_j^{k-1}$;

if $n_{k-1} < i \leqslant j$, then $d_i^k + \ldots + d_j^k = d_{i-n_{k-1}}^{k-1} + \ldots + d_{j-n_{k-1}}^{k-1}$ and (3.5) follows by inductive hypothesis.

Now, using (1.4) we compute $\Phi^{n_k+1}$. We have

$$\Phi^{n_k+1}(d) = \sum_{d_1, \ldots, d_{n_k}} \Phi\big(d - (d_1 + \ldots + d_{n_k})\big)^{-(d_1 + \ldots + d_{n_k})} \Phi(d_1)^{-(d_2 + \ldots + d_{n_k})} \times$$
$$\times \Phi(d_2)^{-(d_3 + \ldots + d_{n_k})} \ldots \Phi(d_{n_k})$$
$$= \sum_{d_1, \ldots, d_{n_k}} t(d - d_1 - \ldots - d_{n_k}) f_{d_1 + \ldots + d_{n_k}} t(d_1) f_{d_2 + \ldots + d_{n_k}} t(d_2) f_{d_3 + \ldots + d_{n_k}} t(d_{n_k}) f_0,$$

where

$$t(d) = \begin{cases} 2^{-j} & \text{if } d = a_j,\ j = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\{d_1^k + \ldots + d_{n_k}^k, d_2^k + \ldots + d_{n_k}^k, \ldots, d_{n_k}^k\} = \boldsymbol{d}. \tag{3.6}$$

As we have seen above, the set $\boldsymbol{d}$ consists of $n_k$ different elements listed in (3.6).

By (3.2) we have

$$\begin{aligned}\langle \Phi^{n_k+1}(d_1^k + \ldots + d_{n_k}^k), F_{\boldsymbol{d}}\rangle &\geqslant t(0)t(d_1^k) \ldots t(d_{n_k}^k) \\ &= 2^{-2^{k-1}} \cdot 2^{-2\cdot 2^{k-2}} \cdot 2^{-3\cdot 2^{k-3}} \cdot \ldots \cdot 2^{-k} \\ &= 2^{-(2^{k-1}+2\cdot 2^{k-2}+3\cdot 2^{k-3}+\ldots+k)} \\ &> 2^{-2(n_k+1)} = 4^{-(n_k+1)},\end{aligned}$$

since, clearly,

$$\begin{aligned}2^{k-1}+2\cdot 2^{k-2}+3\cdot 2^{k-3}+\ldots+k &= (2^k-1)+(2^{k-1}-1)+\ldots+1 \\ &= 2^{k+1}-k.\end{aligned}$$

Consequently,

$$\begin{aligned}\|\Phi^{n_k+1}\|_{L^1(D,B)} \geqslant \|\Phi^{n_k+1}(d_1^k + \ldots + d_{n_k}^k)\|_B &\geqslant \|F_{\boldsymbol{d}}\|_{B^*}^{-1} \cdot 4^{-(n_k+1)} \\ &= \|T_0\|_{PM}^{-n_k} \cdot 4^{-(n_k+1)},\end{aligned}$$

whence

$$\lim_{n\to\infty} \|\Phi^n\|_{L^1(D,B)}^{1/n} = \limsup_{n\to\infty} \|\Phi^n\|_{L^1(D,B)}^{1/n} \geqslant (4\|T_0\|_{PM})^{-1} > 0.$$

**4. Non-symmetry and discontinuity of spectra.** Horst Leptin has shown to the author how the theorem of the previous section can be used to disprove symmetry of $L^1$-algebras of certain groups. His argument is as follows.

Suppose that a group $G$ is a split extension of an Abelian group $N$ by a group $S$. Then

$$L^1(G) = L^1(S, L^1(N)).$$

Suppose further that $L^1(N)$ has a $^*$-subalgebra $K$ which is stable under the action of $S$ on $N$ and $I$ is an $S$-stable $^*$-ideal of $K$ such that $B = K/I$ is radical.

THEOREM (Leptin). *If $L^1(S, B)$ is a non-radical Banach $^*$-algebra, then $L^1(G)$ is non-symmetric.*

Proof. Assume that $L^1(G)$ is symmetric. Then, since

$$L^1(S, B) = L^1(S, K)/L^1(S, I),$$

$L^1(S, B)$ is symmetric because $L^1(S, K)$ is a $^*$-subalgebra of $L^1(S, L^1(N))$. But a symmetric non-radical Banach $^*$-algebra $L^1(S, B)$ has a non-zero $^*$-representation.

On the other hand, by the Blattner-Leptin theorem (cf. [6]), for every *-representation $\pi$ of a Leptin algebra $L^1(S, B)$ with a trivial factor-system, there exist a unitary representation $\pi_1$ of $S$ and a *-representation $\pi_2$ of $B$ such that

$$\pi(f) = \int \pi_1(x)\pi_2\big(f(x)\big)dx.$$

But, since $B$ is radical, $\pi_2 = 0$, whence $\pi = 0$, which is a contradiction.

We conclude with some remarks concerning the groups discussed in Section 3.

The dual group $\hat{D}$ of the group $D$ discussed in Section 3 is, of course, the direct sum of countably many copies of $D$, so isomorphic to $D$. On the other hand,

$$L^1\big(D, A(D)\big) = L^1\big(D, L^1(\hat{D})\big) = L^1(D\hat{D}).$$

It is obvious that $D\hat{D}$ is solvable of class 2 and locally finite. Even more is true: since every element of $D$ is of order 2, every element of $D\hat{D}$ is at most of order 4. Still $L^1(D\hat{D})$ is non-symmetric. This can be also alternatively seen as follows.

As previously, it suffices to show that the Leptin algebra $L^1(D, B)$ is non-symmetric. We have shown that $L^1(D, B)$ contains a function $\psi = \psi^*$ such that the spectral radius $\nu(\psi)$ is greater than zero. On the other hand, since

$$D = \bigcup_n D_n,$$

where $\{D_n\}$ is an increasing sequence of finite groups, the algebra $L^1(D, B)$ contains radical subalgebras $L^1(D_n, B)$ and, if $\psi_n = \psi | D_n$, we have

$$\lim_{n\to\infty} \|\psi - \psi_n\|_{L^1(D,B)} = 0.$$

Consequently, a Hermitian element $\psi$ with $\nu(\psi) > 0$ is a limit of Hermitian elements $\psi_n$ with $\nu(\psi_n) = 0$ and this cannot happen in a symmetric algebra.

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