COLLOQUIUM MATHEMATICUM

VOL. XLI 1979 FASC. 1

*DILATIONS AND GAUGES ON NILPOTENT LIE GROUPS**

BY

J. W. JENKINS (SUNY, ALBANY)

Let L denote a real, finite-dimensional nilpotent Lie algebra. A *family of dilations* $\{\delta_r\}_{r>0}$ on L is a one-parameter group of automorphisms of L given by $\delta_r = r^D$, where D is a diagonalizable operator on L with eigenvalues $\Lambda = \{1 = \lambda_1 < \ldots < \lambda_p\}$. For each $\lambda \in \Lambda$, let

$$L_\lambda = \{X \in L \mid \delta_r X = r^\lambda X\}.$$

Then $L = \oplus L_\lambda$, and for each $\lambda \in \Lambda$ there is a projection P_λ of L onto L_λ. Fix a norm $\|\cdot\|$ on L, so that the various eigenspaces L_λ are orthogonal. Define $|\cdot|$ on L by

$$|X| = \sup_{\lambda \in \Lambda} \|P_\lambda X\|^{1/\lambda}.$$

One easily sees that $|X| = 0$ if and only if $X = 0$, and that $|\delta_r X| = r|X|$. Any non-negative-valued continuous function on L satisfying these latter two conditions is called a *homogeneous gauge*.

Given a relatively compact neighborhood U of zero in L, define τ_U on L by

$$\tau_U(X) = \inf\{n \mid X \in U^n\}, \quad \text{where } U^n = \{X_1 \ldots X_n \mid X_i \in U\},$$

the multiplication being defined by the Campbell-Hausdorff formula. In this note we determine when and how τ_U is related to a homogeneous gauge.

Remarks. For further facts and various applications of homogeneous gauges and the gauge τ_U, see [3]-[5] and [7]-[9].

Although it is known that not all nilpotent Lie algebras admit a family of dilations (cf. [1]), there is currently no classification of those algebras which do admit dilations. In [4], Goodman shows that the graded algebra associated with a nilpotent algebra has dilations, and that these are, "asymptotically", dilations of the original algebra.

* This research was partially supported by a grant from the National Science Foundation Grant # 020-0563A.

Let

$$L = L^{(1)} \supset L^{(2)} \supset \ldots \supset L^{(r)} \supset L^{(r+1)} = \{o\}$$

denote the descending central series of L. Fix a norm $\|\cdot\|$ on L, and define the function $|\cdot|$ on L by

$$|X| = \sup_k \|X\|^{1/k}, \quad \text{where } X = \sum X_k,\ X_k \in L^{(k)}/L^{(k+1)}.$$

Guivarc'h [6] has shown that $|\cdot|$ satisfies condition (i) of Theorem 1.

The author is indebted to Professors R. Goodman and A. Hulanicki for discussions that led to this paper.

Assume henceforth that L has a family of dilations $\{\delta_r\}_{r>0}$, and let L_λ and $|\cdot|$ be as above. For $t > 0$ let $L(t) = \{X \in L \mid |X| \leqslant t\}$ and set $L_\lambda(t) = L_\lambda \cap L(t)$ for each $\lambda \in \Lambda$, $t > 0$.

LEMMA 1. *There is an $a > 0$ such that, for all positive integers n, $\delta_{1/n}(L_1(1))^n \subset L(a)$.*

Proof. Let $k = \dim L_1$, let $\{Y_1, \ldots, Y_k\}$ be a basis for L_1 that is orthonormal with respect to $\|\cdot\|$, and set

$$a' = \sup\left\{\left\|\left[Y_{p_1}, [\ldots[Y_{p_{j-1}}, Y_{p_j}]\ldots]\right]\right\|\right\}.$$

Note that

$$L_1(1) \subset \left\{\sum_{j=1}^{k} c_j Y_j \mid |c_j| \leqslant 1, 1 \leqslant j \leqslant k\right\}.$$

Fix a positive integer n, and let

$$X_i = \sum_{j=1}^{k} c_{ij} Y_j$$

be elements of $L_1(1)$ for $1 \leqslant i \leqslant n$. Now

$$\left\|\left[X_{i_1}, [\ldots[X_{i_{j-1}}, X_{i_j}]\ldots]\right]\right\| \leqslant \sum_{p_1,\ldots,p_j=1}^{k} |c_{i_1p_1}|\ldots|c_{i_jp_j}| \left\|\left[Y_{p_1}, [\ldots[Y_{p_{j-1}}, Y_{p_j}]\ldots]\right]\right\| \leqslant a'k^j.$$

Let $C^{(m)}$ denote the set of all commutators of length m formed using elements of $\{X_1, \ldots, X_n\}$,

$$C_1^{(m)} = \left\{\sum_{i=1}^{p} a_i Z_i \mid |a_i| \leqslant 1,\ Z_i \in C^{(m)}, 1 \leqslant i \leqslant p, p \geqslant 1\right\},$$

$$C = \bigcup_{m=1}^{\infty} C^{(m)}, \quad \text{and} \quad C_1 = \sum_{m=1}^{\infty} \oplus C_1^{(m)}.$$

If

$$Z = \sum_{i=1}^{p} a_i Z_i$$

is an element of C_1, then

$$ZX_j = \sum \frac{(-1)^{k-1}}{k} \frac{1}{p_1!q_1!\dots p_k!q_k!} \frac{1}{p_1+q_1+\dots+p_k+q_k} \times$$
$$\times [\underbrace{Z\dots[Z}_{p_1}, [\underbrace{X_j\dots[X_j}_{q_1}\dots[\underbrace{Z\dots[Z}_{p_k}, [\underbrace{X_j\dots X_j}_{q_k}]\dots]$$
$$= \sum \frac{(-1)^{k-1}}{k} \frac{1}{p_1!\dots q_k!} \frac{1}{p_1+\dots+q_k} \sum a_{i_1}\dots a_{i_r} \times$$
$$\times [\underbrace{Z_{i_1}\dots[Z_{i_j}}_{p_1}, [\underbrace{X_j\dots[X_j}_{q_1}\dots[\underbrace{Z_{i_r-p_k}\dots[Z_{i_r}}_{p_k}, [\underbrace{X_j\dots X_j}_{q_k}]\dots],$$

where $r = p_1+\dots+p_k$, and the second summation in the last expression is over all r-tuples $(i_1, \dots, i_r)$ in $(1, \dots, p)^r$ (cf. [2]). Thus, $ZY_j \in C_1$ if $Z \in C_1$ and $1 \leqslant j \leqslant n$. Therefore, $X_1\dots X_n \in C_1$.

Let

$$X_1\dots X_n = \sum_{j=1}^{p} \beta_j Z_j,$$

where $|\beta_j| \leqslant 1$ and $Z_j \in C$ for $1 \leqslant j \leqslant p$. Then

$$\|P_i(X_1\dots X_n)\| = \Big\| \sum_{Z_j \in C^{(i)}} \beta_j Z_j \Big\| \leqslant \sum_{Z_j \in C^{(i)}} \|Z_J\| \leqslant \sum_{Z_j \in C^{(i)}} a'k^i \leqslant a'k^i n^i$$

since $C^{(i)}$ contains at most n^i terms. Thus

$$|X_1\dots X_n| = \sup_{\lambda \in \Lambda} \|P_\lambda(X_1\dots X_n)\|^{1/\lambda} \leqslant a'kn,$$

and hence

$$[\delta_{1/n}(X_1\dots X_n)| = \frac{1}{n} |(X_1\dots X_n)| \leqslant a'k.$$

Therefore, $\delta_{1/n}(L_1(1))^n \subset L(a'k)$.

LEMMA 2. *Assume that L_1 generates L as a Lie algebra. For any compact neighborhood U of zero there exist $s, t > 0$ such that $L(sn) \subset U^n \subset L(tn)$ for all positive integers n.*

Proof. By Lemma 5.1 of [3], there is a positive integer k such that $(L_1(1))^k$ is a neighborhood of zero. Since L is connected and $L(1)$ is compact, there is an integer m such that $L(1) \subset (L_1(1))^m$. Clearly, $\{(L_1(t))^m \mid t > 0\}$

is a fundamental system of neighborhoods of zero in L, and hence, given U, there is an $s > 0$ such that $(L_1(s))^m \subset U$. Also, since

$$L = \bigcup_{t>0} (L_1(t))^m,$$

there is an $r > 0$ such that $U \subset (L_1(t))^m$.

Now, for all positive integers n,

$$L(sn) = \delta_n L(s) \subset \delta_n (L_1(s))^m = (L_1(ns))^m \subset (L_1(s))^{nm}.$$

Thus, by Lemma 1,

$$(L_1(r))^{mn} = \delta_r (L_1(1))^{mn} \subset \delta_r L(mna) = L(rmna).$$

Thus, letting $t = arm$, we have

$$L(sn) \subset (L_1(s))^{mn} \subset U^n, \qquad L \subset L_1(r)^{mn} \subset L(tn)$$

for all positive integers n.

LEMMA 3. *If the Lie algebra generated by L_1 is distinct from L, then, given any compact neighborhood U of zero and any positive integer k, there is an N such that $L(n) \sim U^{kn} \neq \emptyset$ for all integers $n \geqslant N$.*

Proof. Let L' denote the Lie algebra generated by L_1, and let $B = \{X_0, \ldots, X_p\}$ be a basis for L consisting of eigenvectors for $\{\delta_r\}$. Under the assumption on L_1, there are $X_i \in B$ such that $X_i \notin L'$. Let

$$\lambda_0 = \inf\{\lambda L_\lambda \sim L' \neq \emptyset\}.$$

By relabeling, if necessary, we may assume that $X_0 \in L_{\lambda_0} \sim L'$. Let L_0 denote the span of $\{X_1, \ldots, X_p\}$. If $X_i \in L_0$, then either $X_i \in L'$ or $X_i \in L_\lambda$, where $\lambda \geqslant \lambda_0$. It follows that $[X_i, X_j] \in L_0$ if $X_i, X_j \in L_0$, and hence L_0 is a subalgebra. Also, if $X_i \in L_0$ and if

$$[X_i, X_0] = aX_0 + \sum_{k\neq 0} \beta_k X_k,$$

then $a = 0$, since $P_{\lambda_0}([X_i, X_0]) = 0$. Thus, as a multiplicative group, L_0 is normal in L.

Fix a compact neighborhood U of zero and a positive integer k. Pick a compact neighborhood U_0 of zero in L_0 and an $a > 0$ such that

$$U^k \subset \bigcup_{|t|\leqslant a} (tX_0)\, U_0.$$

Then

$$\begin{aligned} U^{kn} &\subset \bigcup_{|t_i|\leqslant a} (t_1 X_0)\, U_0 \ldots (t_n X_0)\, U_0 \\ &= \bigcup_{|t_i|\leqslant a} (t_1 X_0) \ldots (t_n X_0)\, U_0^{(t_2X_0)\ldots(t_nX_0)} \ldots U_0^{t_nX_0} U_0, \end{aligned}$$

where $V^X = \{(-X)\,YX \mid Y \in V\}$. Since $U_0 \subset L_0$ for all $Y \in L$ and since

$$(t_1 X_0) \ldots (t_n X_0) = (t_1 + \ldots + t_n) X_0,$$

we have

$$U^{kn} \subset \bigcup_{|t| \leqslant an} (tX_0)L_0.$$

If

$$X_0 = \sum_{\lambda \in \Lambda} X_\lambda, \qquad \text{where } X_\lambda \in L_\lambda,$$

and if $\{t_n\}$ is any sequence satisfying $|t_n| \leqslant an$, then, since $X_1 = 0$,

$$\lim_{n\to\infty} \delta_{1/n}(t_n X_0) = \lim_{n\to\infty} \sum_{\lambda \in \Lambda} t_n n^{-\lambda} X_\lambda = 0.$$

Thus, given $\varepsilon > 0$, there is an N such that

$$\delta_{1/n}(U^{kn}) \subset \bigcup_{|t| \leqslant an} (t\,\delta_{1/n}X_0)(\delta_{1/n}L_0) \subset \bigcup_{|t| < \varepsilon} (tX_0)L_0$$

for all $n \geqslant N$. Choosing ε sufficiently small we have

$$L(1) \sim \bigcup_{|t| \leqslant \varepsilon} (tX_0)L_0 \neq \emptyset,$$

and thus

$$L(n) \sim U^{kn} = \delta_n\big(L(1) \sim \delta_{1/n} U^{kn}\big) \supset \delta_n\big(L(1) \sim \bigcup_{|t| \leqslant \varepsilon} (tX_0)L_0\big) \neq \emptyset.$$

THEOREM 4. *Let U be a compact neighborhood of zero in L and let $|\cdot|$ denote any homogeneous gauge on L. The following statements are equivalent:*

(i) *given $\varepsilon > 0$ there exist $a, b > 0$ such that if $|X| > \varepsilon$, then*

$$a|X| \leqslant \tau_U(X) \leqslant b|X|;$$

(ii) *L_1 generates L as a Lie algebra.*

Proof. Goodman [4] has shown that, given homogeneous gauges $|\cdot|_1$ and $|\cdot|_2$, there exist $a', b' > 0$ such that

$$a'|X|_1 \leqslant |X|_2 \leqslant b'|X|_1 \qquad \text{for all } X \in L.$$

Thus, we may assume that $|\cdot|$ denotes the gauge defined previously.

Assume that L_1 generates L. Given U and $\varepsilon > 0$, choose $s, t > 0$ such that $L(sn) \subset U^n \subset L(tn)$ for all positive integers n. Suppose that X is given with $|X| > \varepsilon$. If $s(n-1) < |X| \leqslant sn$, then $x \in L(sn) \subset U^n$, so

$$\tau_U(X) \leqslant n < s^{-1}|X| + 1 < (s^{-1} + \varepsilon^{-1})|X|.$$

If $\tau_U(X) = m$, then $X \in U^m \subset L(tm)$, and thus

$$|X| \leqslant tm = t\tau_U(X).$$

Therefore

$$t^{-1}|X| \leqslant \tau_U(X) \leqslant (s^{-1} + \varepsilon^{-1})|X|.$$

Assume now that L_1 does not generate L. Fix a compact neighborhood U of zero. By Lemma 3, for each positive integer k there is an $X_k \in L(n) \sim U^{kn}$ for n sufficiently large. Thus $\tau_U(X_k) \geqslant kn \geqslant k|X_k|$, contradicting (i).

Trivial examples show that L_1 need not generate L. For example, if L is an Abelian Lie algebra and D is given by a diagonal matrix with $1 = d_{11} \leqslant d_{22} \leqslant \ldots \leqslant d_{nn}$, then setting $\delta_r = r^D$ defines a family of dilations of L. The corresponding L_1 generates L if and only if $d_{11} = d_{22} = \ldots = d_{nn}$.

This example does not preclude the possibility that if L admits dilations, then, for some dilations of L, the corresponding L_1 generates L. The following example shows that this need not be the case.

Let L be the real Lie algebra with basis $X_1, \ldots, X_6$ and relations

$$[X_1, X_2] = X_4, \quad [X_2, X_3] = X_5,$$

$$[X_1, X_3] = X_6 = [X_1, X_4] = [X_2, X_4]$$

and all other commutators of basis elements equal to zero. This algebra can be realized as a subalgebra of the upper triangular (4×4)-matrices with

$$X_1 = (1, 1, 1, 1, 1, 0), \quad X_2 = (1, 0, -1, 1, 1, 0),$$

$$X_3 = (1, 0, 1, 0, -1, 0),$$

where we have written $(a_1, \ldots, a_6)$ for

$$\begin{pmatrix} 0 & a_1 & a_4 & a_6 \\ 0 & 0 & a_2 & a_5 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $\{\delta_r\}_{r>0}$ is a one-parameter group of dilations of L for which the corresponding L_1 generates L, then we must have

$$\delta_r X_1 = rX_1, \quad \delta_r X_2 = rX_2, \quad \delta_r X_3 = rX_3,$$

since $\{X_1, X_2, X_3\}$ is a basis for $L \sim [L, L]$. However, since

$$X_6 = [X_1, X_3] = [X_1, X_4] = \big[X_1[X_1, X_2]\big],$$

and since δ_r is an automorphism of L, we have

$$\delta_r X_6 = \delta_r[X_1, X_3] = [\delta_r X_1, \delta_r X_3] = r^2 X_6$$

and

$$\delta_r X_6 = \delta_r\big[X_1[X_1 X_2]\big] = r^3 X_6.$$

This contradiction shows that L_1 cannot generate L. If one defines δ_r by setting $\delta_r = r^D$, where D is the diagonal matrix having

$$d_{11} = d_{22} = 1, \quad d_{33} = d_{44} = 2, \quad d_{55} = d_{66} = 3$$

with respect to the basis $\{X_1, \ldots, X_6\}$, then $\{\delta_r\}_{r>0}$ is a family of dilations of L.

REFERENCES

[1] J. L. Dyer, *A nilpotent Lie algebra with nilpotent automorphism group*, Bulletin of the American Mathematical Society 76 (1970), p. 52-56.

[2] E. B. Dynkin, *Normed Lie algebras and analytic groups*, American Mathematical Society Translations 97 (1953), p. 470-534.

[3] G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Arkiv för Matematik 13 (1975), p. 161-208.

[4] R. Goodman, *Filtrations and asymptotic automorphisms on nilpotent Lie groups* (to appear).

[5] — *Filtrations and canonical coordinates on nilpotent Lie groups* (to appear).

[6] Y. Guivarc'h, *Croissance polynomiale et périodes des fonctions harmoniques*, Bulletin de la Société Mathématique de France 101 (1973), p. 333-379.

[7] A. Hulanicki, *Commutative subalgebra of $L^1(G)$ associated with a subelliptic operator on a Lie group G*, Bulletin of the American Mathematical Society 81 (1975), p. 121-124.

[8] — *Subalgebra of $L^1(G)$ associated with Laplacian on a Lie group* (to appear).

[9] A. W. Knapp and E. M. Stein, *Intertwining operators for semisimple groups*, Annals of Mathematics 93 (1971), p. 489-578.

STATE UNIVERSITY OF NEW YORK AT ALBANY

Reçu par la Rédaction le 30. 4. 1977