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# *ON CONFORMALLY RECURRENT RICCI-RECURRENT MANIFOLDS*

BY

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**1. Introduction.** Let $M$ be a Riemannian manifold with a (possibly indefinite) metric $g$.

A tensor field $T^{i_1\ldots i_p}{}_{j_1\ldots j_q}$ of type $(p, q)$ on $M$ will be called *recurrent* if

$$T^{h_1\ldots h_p}{}_{t_1\ldots t_q} T^{i_1\ldots i_p}{}_{j_1\ldots j_q,k} = T^{h_1\ldots h_p}{}_{t_1\ldots t_q,k} T^{i_1\ldots i_p}{}_{j_1\ldots j_q}, \tag{1}$$

where the comma denotes covariant differentiation with respect to $g$.

Relation (1) states that at any point $x \in M$ such that $T_x \neq 0$ there exists a (unique) covariant vector $u$ (called the *recurrence vector* of $T$) which satisfies the condition

$$T^{i_1\ldots i_p}{}_{j_1\ldots j_q,k}(x) = u_k T^{i_1\ldots i_p}{}_{j_1\ldots j_q}(x). \tag{2}$$

A Riemannian manifold $(M, g)$ will be called *recurrent* [9] (*Ricci-recurrent* [5]) if its curvature tensor (Ricci tensor) is recurrent.

Throughout this paper we assume that the Ricci tensor of a Ricci-recurrent manifold is not parallel.

According to Adati and Miyazawa [1], an $n$-dimensional $(n \geq 4)$ Riemannian manifold $(M, g)$ will be called *conformally recurrent* if its Weyl conformal curvature tensor

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}) + \\ + \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{ik}g_{hj}) \tag{3}$$

is recurrent.

If $C_{hijk,l} = 0$ everywhere on $M$ and $\dim M \geq 4$, then $(M, g)$ is said to be *conformally symmetric* [2].

Clearly, the class of conformally recurrent manifolds contains all conformally symmetric as well as all recurrent manifolds of dimension $n \geq 4$.

The existence of essentially conformally recurrent manifolds, i.e., of conformally recurrent manifolds which are neither conformally symmetric nor recurrent can be stated as follows (see [8], Lemmas 2-3):

THEOREM. *Let $R^n$ be endowed with the metric $g_{ij}$ given by*

$$(4)\qquad \begin{aligned} g_{ij}dx^i dx^j &= Q(dx^1)^2 + k_{\lambda\mu}dx^\lambda dx^\mu + 2dx^1 dx^n, \\ Q &= (Ap_{\lambda\mu} + k_{\lambda\mu})x^\lambda x^\mu, \end{aligned}$$

*where $i, j = 1, 2, \ldots, n$, $\lambda, \mu = 2, 3, \ldots, n-1$, $[k_{\lambda\mu}]$ is a symmetric and non-singular matrix, $[p_{\lambda\mu}]$ is a symmetric matrix satisfying $[p_{\lambda\mu}] \neq (n-2)^{-1}[k_{\lambda\mu}]$ and $k^{\lambda\mu}p_{\lambda\mu} = 1$ with $[k^{\lambda\mu}] = [k_{\lambda\mu}]^{-1}$, and $A$ is a non-constant function of $x^1$ only.*

*Then $(R^n, g)$ is an essentially conformally recurrent manifold which is, moreover, Ricci-recurrent (with a non-parallel Ricci tensor) and satisfies the condition*

$$(5)\qquad C_{hijk,lm} - C_{hijk,ml} = 0.$$

The purpose of this paper is to obtain a local metric form for a conformally recurrent manifold $M_n$ ($n > 4$) which is simultaneously Ricci-recurrent and satisfies condition (5).

All manifolds under consideration are connected Hausdorff manifolds. The metrics are not assumed to be definite.

**2. Preliminaries.** In the sequel we shall need the following lemmas:

LEMMA 1. *The Weyl conformal curvature tensor satisfies the well-known relations*

$$(6)\qquad C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkhi}, \qquad C^r{}_{rij} = C^r{}_{irk} = C^r{}_{ijr} = 0,$$

$$(7)\qquad C_{hijk} + C_{hjki} + C_{hkij} = 0,$$

$$(8)\qquad C^r{}_{ijk,r} = \frac{n-3}{n-2}\left[(R_{ij,k} - R_{ik,j}) - \frac{1}{2(n-1)}(R_{,k}g_{ij} - R_{,j}g_{ik})\right].$$

LEMMA 2 ([1], equation (3.7), and [4], p. 91). *The Weyl conformal curvature tensor satisfies the relation*

$$(9)\qquad \begin{aligned} C_{hijk,l} + C_{hikl,j} + C_{hilj,k} = \frac{1}{n-3}(&g_{ik}C^r{}_{hjl,r} + g_{hj}C^r{}_{ikl,r} + g_{il}C^r{}_{hkj,r} + \\ &+ g_{hk}C^r{}_{ilj,r} + g_{ij}C^r{}_{hlk,r} + g_{hl}C^r{}_{ijk,r}). \end{aligned}$$

LEMMA 3 (see [10], Lemma 1, and [9], p. 153). *The curvature tensor of an arbitrary Riemannian manifold satisfies the identity*

$$R_{hijk,lm} - R_{hijk,ml} + R_{jklm,hi} - R_{jklm,ih} + R_{lmhi,jk} - R_{lmhi,kj} = 0.$$

LEMMA 4. *The Ricci tensor of a Ricci-recurrent manifold satisfies the equations*

$$R_{ab}R_{ij,l} = R_{ab,l}R_{ij}, \tag{10}$$

$$R_{,l}R_{ij} = RR_{ij,l}. \tag{11}$$

Proof. Relations (10) and (11) are immediate consequences of (1).

LEMMA 5. *Let $M$ be a conformally recurrent Ricci-recurrent manifold. If condition* (5) *is satisfied, then the equation*

$$R_{ij,lm} - R_{ij,ml} = R_{ri}R^r{}_{jlm} + R_{rj}R^r{}_{ilm} = 0 \tag{12}$$

*holds on $M$.*

Proof. It is sufficient to prove (12) at points where $R_{ij} \neq 0$.

As an immediate consequence of (3) and (5) we get

$$R_{hijk,lm} - R_{hijk,ml} = \frac{1}{n-2}\,[g_{ij}(R_{hk,lm} - R_{hk,ml}) - g_{ik}(R_{hj,lm} - R_{hj,ml}) +$$

$$+ g_{hk}(R_{ij,lm} - R_{ij,ml}) - g_{hj}(R_{ik,lm} - R_{ik,ml})],$$

whence, in view of (2) and Lemma 3, we obtain

$$A_{lm}D_{hijk} + A_{hi}D_{jklm} + A_{jk}D_{lmhi} = 0,$$

where $D_{hijk} = g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}$, $A_{ij} = a_{i,j} - a_{j,i}$, and $a_j$ denotes the recurrence vector of $R_{ij}$.

But from the last relation, since $R_{ij} \neq 0$, we get (see [10], Lemma 2) $A_{ij} = 0$. Using now (2) and the Ricci identity, we obtain easily (12). The lemma is thus proved.

Remark 1. Equation (12) follows from the proof of Adati's and Miyazawa's Lemma (2.1) of [1]. Since the definition of recurrency in [1] differs slightly from the ours, we have included the proof of (12) for completeness.

LEMMA 6. *Let $M$ be a Ricci-recurrent manifold satisfying* (12). *Then the relation*

$$R_{ri}R^r{}_j = \tfrac{1}{2}RR_{ij} \tag{13}$$

*holds on $M$.*

Proof. This lemma was proved for the open subset $U$ of $M$ where $R_{hk,l} \neq 0$ (see [6], Lemma 2). But outside of $U$, $R_{ij,k}$ vanishes and, therefore, the tensor $B_{ij} = R_{ri}R^r{}_j - \frac{1}{2}RR_{ij}$ is parallel on $M$. Since $B_{ij}$ vanishes on $U$, it vanishes everywhere on $M$. This completes the proof.

LEMMA 7. *Let $M$ be a conformally recurrent Ricci-recurrent manifold. If condition* (5) *is satisfied, then the relations*

$$(R_{ab}R_{dpqt,l} - R_{ab,l}R_{dpqt})R^{rs}C_{rijs} = 0, \tag{14}$$

(15) $$R(R_{ab}R_{dpqt,l}-R_{ab,l}R_{dpqt}) = 0$$

*hold on M.*

Proof. By a direct calculation, in view of (12) and (13), we get

(16)
$$R_{rm}C^r{}_{ijk}+R_{rl}C^r{}_{mjk} = \frac{3-n}{2(n-1)(n-2)}R(g_{ij}R_{mk}-g_{ik}R_{mj}+g_{mj}R_{ik}-g_{mk}R_{ij}),$$

whence, by contraction with $g^{mk}$,

(17) $$R^{rs}C_{rijs} = \frac{3-n}{2(n-1)(n-2)}R(Rg_{ij}-nR_{ij}).$$

Differentiating (17) covariantly and making use of (10), (11), and (17), we obtain

$$R_{ab}R^{rs}C_{rijs,l} = R^{rs}C_{rijs}R_{ab,l},$$

whence, because of (1),

(18) $$(R_{ab}C_{dpqt,l}-R_{ab,l}C_{dpqt})R^{rs}C_{rijs} = 0.$$

But as an immediate consequence of (3), (10), and (11) we get

(19) $$R_{ab}C_{dpqt,l}-R_{ab,l}C_{dpqt} = R_{ab}R_{dpqt,l}-R_{ab,l}R_{dpqt}.$$

The last result, together with (18), yields (14).

Now, in view of (14), equation (17) implies

$$R(Rg_{ij}-nR_{ij})(R_{ab}R_{dpqt,l}-R_{ab,l}R_{dpqt}) = 0,$$

whence, transvecting with $R^i{}_k$ and making use of (13), we obtain

$$RR_{kj}(R_{ab}R_{dpqt,l}-R_{ab,l}R_{dpqt}) = 0,$$

which, evidently, completes the proof.

LEMMA 8. *Let M be a conformally recurrent manifold. If M is Ricci-recurrent and satisfies* (5), *then the relation*

(20) $$(R_{ab}R_{dpqt,l}-R_{ab,l}R_{dpqt})T_{hlmijk} = 0$$

*holds on M, where*

$$\begin{aligned}T_{hlmijk} = {} & g_{hl}R_{rm}C^r{}_{ijk}-g_{hm}R_{rl}C^r{}_{ijk}-g_{il}R_{rm}C^r{}_{hjk}+\\ & +g_{im}R_{rl}C^r{}_{hjk}+g_{jl}R_{rm}C^r{}_{khi}-g_{jm}R_{rl}C^r{}_{khi}+\\ & +g_{kl}R_{rm}C^r{}_{jih}-g_{km}R_{rl}C^r{}_{jih}+R_{hl}C_{mijk}-R_{hm}C_{lijk}+\\ & +R_{il}C_{hmjk}-R_{im}C_{hljk}+R_{jl}C_{himk}-R_{jm}C_{hilk}+\\ & +R_{kl}C_{hijm}-R_{km}C_{hijl}.\end{aligned}$$

Proof. Applying the Ricci identity to (5), we get

$$C_{rijk}R^r{}_{hlm}+C_{hrjk}R^r{}_{ilm}+C_{hirk}R^r{}_{jlm}+C_{hijr}R^r{}_{klm}=0, \tag{21}$$

whence, by covariant differentiation, the use of (1), (21), (3), (6), and Lemma 4, we obtain

$$R_{qt}C_{abcd,p}(C_{rijk}C^r{}_{hlm}+C_{hrjk}C^r{}_{ilm}+C_{hirk}C^r{}_{jlm}+C_{hijr}C^r{}_{klm})+ \\ +\frac{1}{n-2}C_{abcd}\left(T_{hlmijk}-\frac{R}{n-1}D_{hlmijk}\right)R_{qt,p}=0, \tag{22}$$

where

$$D_{hlmijk}=g_{hl}C_{mijk}-g_{hm}C_{lijk}+g_{il}C_{hmjk}-g_{im}C_{hljk}+ \\ +g_{jl}C_{himk}-g_{jm}C_{hilk}+g_{kl}C_{hijm}-g_{km}C_{hijl}.$$

On the other hand, substituting into (21) the expression

$$R^h{}_{ijk}=C^h{}_{ijk}+\frac{1}{n-2}(g_{ij}R^h{}_k-g_{ik}R^h{}_j+\delta^h_kR_{ij}-\delta^h_jR_{ik})- \\ -\frac{R}{(n-1)(n-2)}(\delta^h_kg_{ij}-\delta^h_jg_{ik}),$$

differentiating covariantly, and using (1) and Lemma 4, we get

$$2R_{qt}(C_{rijk}C^r{}_{hlm}+C_{hrjk}C^r{}_{ilm}+C_{hirk}C^r{}_{jlm}+C_{hijr}C^r{}_{klm})C_{abcd,p}+ \\ +\frac{1}{n-2}(R_{qt}C_{abcd,p}+R_{qt,p}C_{abcd})\left(T_{hlmijk}-\frac{R}{n-1}D_{hlmijk}\right)=0.$$

But the last relation, together with (22), implies

$$(R_{qt}C_{abcd,p}-R_{qt,p}C_{abcd})\left(T_{hlmijk}-\frac{1}{n-1}RD_{hlmijk}\right)=0,$$

which, in view of (19) and (15), leads immediately to our assertion. The lemma is proved.

LEMMA 9. *Let $M$ be a conformally recurrent manifold. If $M$ is Ricci-recurrent and condition* (5) *is satisfied, then the relations*

$$(R_{ab}R_{dpqt,l}-R_{ab,l}R_{dpqt})R_{rm}C^r{}_{ijk}=0, \tag{23}$$

$$(R_{ab}R_{dpqt,l}-R_{ab,l}R_{dpqt})(R_{hl}C_{mijk}-R_{hm}C_{lijk}+R_{il}C_{hmjk}-R_{im}C_{hljk}+ \\ +R_{jl}C_{himk}-R_{jm}C_{hilk}+R_{kl}C_{hijm}-R_{km}C_{hijl})=0 \tag{24}$$

*hold on $M$.*

Proof. Contracting (20) with $g^{hl}$ and using (6), (7), (14), and (15), we get

$$(25)\qquad (R_{ab}R_{dpqt,l}-R_{ab,l}R_{dpqt})[(n-2)R_{rm}C^r{}_{ijk}+R_{rj}C^r{}_{imk}+ \\ +R_{rk}C^r{}_{ijm}+R_{ri}C^r{}_{mjk}] = 0.$$

On the other hand, as a consequence of (15) and (16), we have

$$(R_{ab}R_{dpqt,l}-R_{ab,l}R_{dpqt})R_{rm}C^r{}_{ijk} = -(R_{ab}R_{dpqt,l}-R_{ab,l}R_{dpqt})R_{ri}C^r{}_{mjk}.$$

The last result, together with (25) and (7), leads immediately to (23). Relation (24) follows now from (20) and (23). The lemma is thus proved.

LEMMA 10. *Let $M$ be a Riemannian manifold whose Ricci tensor satisfies*

$$(26)\qquad R_{hl}C_{mijk}-R_{hm}C_{lijk}+R_{il}C_{hmjk}-R_{im}C_{hljk}+ \\ +R_{jl}C_{himk}-R_{jm}C_{hilk}+R_{kl}C_{hijm}-R_{km}C_{hijl} = 0.$$

*Suppose, moreover, that $x \in M$ and* $\operatorname{rank} R_{ij}(x) > 1$.

*Then the Weyl conformal curvature tensor on some neighbourhood of $x$ is of the form*

$$(27)\qquad C_{hijk} = S(R_{ij}R_{hk}-R_{hj}R_{ik}).$$

Proof. This lemma was proved for conformally symmetric manifolds (see [3], Theorem 3). But its proof, as one can easily verify, requires only relations (6), (7), and (26). Therefore, (27) remains true for an arbitrary Riemannian manifold satisfying (26).

LEMMA 11. *Let $M_n$ $(n > 4)$ be a conformally recurrent Ricci-recurrent manifold whose Weyl conformal curvature tensor satisfies* (5). *Suppose that at $x \in M_n$ the conditions*

$$(28)\qquad R_{ab}R_{dpqt,l}-R_{ab,l}R_{dpqt} \neq 0,$$

$$(29)\qquad \operatorname{rank} R_{ij} > 1$$

*hold.*

*Then* $R_{ij,l}(x) = 0$.

Proof. By (24), (27), and (23), we obtain on some neighbourhood $U$ of $x$

$$(30)\qquad C_{rijk}C^r{}_{hlm} = 0.$$

Since (28) holds, in each neighbourhood of $x$ there exists a point $y$ such that $C_{hijk}(y) \neq 0$. For otherwise, $C_{hijk}$ as well as $C_{hijk,l}$ would vanish on some neighbourhood $W$ of $x$ and, consequently, the recurrence condition (1) would be satisfied for $R_{hijk}$ on $W$, a contradiction.

Now, in view of (2) and (9), we get at $y$

$$b_l C_{hijk} + b_j C_{hikl} + b_k C_{hilj} = \frac{1}{n-3}(g_{ik} b_r C^r{}_{hjl} + g_{hj} b_r C^r{}_{ikl} + g_{il} b_r C^r{}_{hkj} + \\ + g_{hk} b_r C^r{}_{ilj} + g_{ij} b_r C^r{}_{hlk} + g_{hl} b_r C^r{}_{ijk}),$$

where $b_j$ denotes the recurrence vector of $C_{hijk}$.

Transvecting the last relation with $C^l{}_{ptm}$ and using (30), we obtain for $y \in U$

$$b_r C^r{}_{ptm} C_{hijk} = \frac{1}{n-3}(b_r C^r{}_{hkj} C_{iptm} + b_r C^r{}_{ijk} C_{hptm}),$$

which, by a further transvection with $b^h$, yields

$$b_r C^r{}_{ptm} b_s C^s{}_{ijk} = \frac{1}{n-3} b_s C^s{}_{ijk} b_r C^r{}_{ptm}.$$

Hence $b_r C^r{}_{ijk} = 0 = C^r{}_{ijk,r}$. But the last result, in view of (15), reduces (8) to the form $R_{ij,k}(y) = R_{ik,j}(y)$. If now $R_{ij,k}(x)$ were non-zero, then, by (2), we would have $\operatorname{rank} R_{ij}(y) = 1$ for some set of $y$'s from $U$, a contradiction. The lemma is thus proved.

In the sequel, we shall often assume the following hypothesis:

(31) $(M, g)$ is an $n$-dimensional $(n > 4)$ conformally recurrent Ricci-recurrent manifold satisfying (5), and $x$ is a point of $M$ such that the conditions

(32) $$R_{ij} \neq 0, \quad R_{ij,k} \neq 0$$

and (28) hold at $x$.

Remark 2. For the metric (4), as one can easily verify [8], conditions (28) and (32) are equivalent to: $A \neq 2-n$, $\partial_1 A \neq 0$.

PROPOSITION 1. *Under hypothesis* (31), $\operatorname{rank} R_{ij}(x) = 1$ *and* $M$ *admits a non-trivial null parallel vector field on some neighbourhood of* $x$.

Proof. The first part of our assertion follows immediately from (32) and Lemma 11.

Since (2) and

(33) $$R_{ij} = e c_i c_j, \quad |e| = 1$$

hold, we have

$$(c_{i,k} - \tfrac{1}{2} a_k c_i) c_j + (c_{j,k} - \tfrac{1}{2} a_k c_j) c_i = 0.$$

Hence

(34) $$c_{j,k} = \tfrac{1}{2} a_k c_j.$$

On the other hand, as a consequence of (2) and (12), we get $a_{i,j} = a_{j,i}$. Therefore, there exists a function $a$ such that $a_{,j} = a_j$. Now, in view of (34), it is easy to verify that the vector field $B_j = \exp(-\frac{1}{2}a)c_j$ is parallel. Moreover, because of (15) and (33), $B_j$ is null. This completes the proof.

LEMMA 12. *Under hypothesis* (31), *the equation*

$$(35) \qquad R_{hl}R_{mijk} - R_{hm}R_{lijk} + R_{il}R_{hmjk} - R_{im}R_{hljk} + \\ + R_{jl}R_{himk} - R_{jm}R_{hilk} + R_{kl}R_{hijm} - R_{km}R_{hijl} = 0$$

*holds on some neighbourhood of* $x$.

Proof. As a consequence of (15), (20), and (22), we get

$$C_{abcd,p}(C_{rijk}C^r{}_{hlm} + C_{hrjk}C^r{}_{ilm} + C_{hirk}C^r{}_{jlm} + C_{hijr}C^r{}_{klm}) = 0$$

on some neighbourhood $U \ni x$. For the open subset of $U$ (if it exists) where $C_{hijk}$ is parallel, (35) follows immediately from (24) of [7].

Thus, we can assume that we have on $U$

$$(36) \qquad C_{rijk}C^r{}_{hlm} + C_{hrjk}C^r{}_{ilm} + C_{hirk}C^r{}_{jlm} + C_{hijr}C^r{}_{klm} = 0.$$

On the other hand, using (12) and (33), we obtain

$$c_i c_r R^r{}_{jlm} + c_j c_r R^r{}_{ilm} = 0.$$

Hence $c_r R^r{}_{jlm} = 0$ and, consequently, $R_{ri}R^r{}_{jlm} = 0$.

But the last result, in view of (33), (23), and

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}),$$

yields

$$(37) \qquad C_{rijk}C^r{}_{hlm} = R_{rijk}R^r{}_{hlm} - \frac{1}{n-2}(R_{ij}R_{khlm} - R_{ik}R_{jhlm} + \\ + R_{hl}R_{mijk} - R_{hm}R_{lijk}) + \frac{1}{(n-2)^2}(g_{km}R_{ij}R_{hl} - g_{jm}R_{ik}R_{hl} + g_{jl}R_{ik}R_{hm} - \\ - g_{lk}R_{ij}R_{hm}).$$

Since

$$0 = R_{hijk,lm} - R_{hijk,ml} = R_{rijk}R^r{}_{hlm} + R_{hrjk}R^r{}_{ilm} + R_{hirk}R^r{}_{jlm} + R_{hijr}R^r{}_{klm},$$

which follows easily from (3), (5), and (12), relation (36), together with (37) and (33), leads to (35). The last remark completes the proof.

PROPOSITION 2. *Under hypothesis* (31), *the curvature tensor takes on some neighbourhood of* $x$ *the form*

$$(38) \qquad R_{jkhm} = c_h c_k S_{mj} - c_h c_j S_{mk} + c_m c_j S_{hk} - c_m c_k S_{hj},$$

*where* $S_{ij} = S_{ji} = v^r v^s R_{rijs}$ *and* $v^r c_r = 1$.

Proof. Since equations (33) and (35) hold, we have (see [7], Lemma 4)

$$c_l R_{jkhm} + c_h R_{jkml} + c_m R_{jklh} = 0.$$

Now, with the help of the last result, we can follow step by step a proof of Walker (see [10], p. 45, and [9], p. 155) to obtain (38). This completes the proof.

**3. Local structure theorem.** We are now in a position to prove our main result. In this section each Latin index runs over $1, 2, \ldots, n$, and each Greek index over $2, 3, \ldots, n-1$.

THEOREM. (i) *Let* $(M, g)$ *be an* $n$*-dimensional* $(n > 4)$ *conformally recurrent Ricci-recurrent manifold satisfying* (5). *Suppose, moreover, that at* $x \in M$ *conditions* (28), (32), *and* $C_{hijk} \neq 0$ *hold. Then there exists a coordinate system* $x^1, x^2, \ldots, x^n$ *on a neighbourhood of* $x$ *such that the metric of* $M$ *takes the form*

$$g_{ij} dx^i dx^j = Q (dx^1)^2 + k_{\lambda\mu} dx^\lambda dx^\mu + 2 dx^1 dx^n, \tag{39}$$

$$\partial_\lambda \partial_\mu Q = A k_{\lambda\mu} + B c_{\lambda\mu}, \tag{40}$$

*where* $[k_{\lambda\mu}]$ *is a symmetric and non-singular matrix,* $[c_{\lambda\mu}]$ *is a symmetric and non-zero matrix satisfying* $k^{\alpha\beta} c_{\alpha\beta} = 0$ *with* $[k^{\lambda\mu}] = [k_{\lambda\mu}]^{-1}$, *and* $A, B$ *are functions not depending on* $x^n$ *such that* (40) *as well as*

$$A \neq 0 \neq B, \quad \partial_j A \neq 0, \quad A \partial_j B - B \partial_j A \neq 0 \tag{41}$$

*are satisfied.*

(ii) *Let* $R^n$ $(n \geqslant 4)$ *be endowed with the metric* $g_{ij}$ *given by* (39), *where* $[k_{\lambda\mu}]$ *and* $[c_{\lambda\mu}]$ *are as above, and* $A, B$ *are non-constant functions satisfying* (40), *independent of* $x^n$, *and such that* $A \partial_j B - B \partial_j A$ *does not indentically vanish. Then* $(R^n, g)$ *is an essentially conformally recurrent Ricci-recurrent manifold whose Weyl conformal curvature tensor satisfies* (5).

Proof. From Walker's considerations it follows ([9], p. 176-179, and [10], p. 51-54) that if a Riemannian manifold admits on a neighbourhood $U$ of a point $x$ a non-trivial null parallel vector field $B_j = f c_j$ and on $U$ the curvature tensor is of the form (38), then one can choose coordinates so that the metric can be written as

$$g_{ij} dx^i dx^j = Q (dx^1)^2 + k_{\lambda\mu} dx^\lambda dx^\mu + 2 dx^1 dx^n, \tag{42}$$

where $k_{\lambda\mu} = k_{\mu\lambda}$ are constants, $\det[k_{\lambda\mu}] \neq 0$, and $Q$ is a function independent of $x^n$.

As one can easily verify,

$$[g^{ij}] = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & & & & 0 \\ \vdots & & k^{\lambda\mu} & & \vdots \\ 0 & & & & 0 \\ 1 & 0 & \dots & 0 & -Q \end{bmatrix}$$

and, in the metric (42), the only Christoffel symbols not identically zero are

$$\left\{ {\lambda \atop 11} \right\} = -\frac{1}{2} k^{\lambda\omega} Q_{.\omega}, \quad \left\{ {n \atop 11} \right\} = \frac{1}{2} Q_{.1}, \quad \left\{ {n \atop 1\lambda} \right\} = \frac{1}{2} Q_{.\lambda},$$

where the dot denotes partial differentiation with respect to coordinates.

Moreover, in view of the formula

$$R_{hijk} = \frac{1}{2}(g_{hk.ij} + g_{ij.hk} - g_{hj.ik} - g_{ik.hj}) + g_{pq} \left\{ {p \atop hk} \right\} \left\{ {q \atop ij} \right\} - g_{pq} \left\{ {p \atop hj} \right\} \left\{ {q \atop ik} \right\},$$

it follows ([9], p. 179) that the only components $R_{hijk}$ not identically zero are those related to $R_{1\lambda\mu 1} = \frac{1}{2} Q_{.\lambda\mu}$.

It can be also found that

$$R_{11} = \frac{1}{2} k^{\alpha\beta} Q_{.\alpha\beta}, \quad R_{11,1} = \frac{1}{2} k^{\alpha\beta} Q_{.\alpha\beta 1}, \quad R_{11,\gamma} = \frac{1}{2} k^{\alpha\beta} Q_{.\alpha\beta\gamma},$$

$$R_{1\lambda\mu 1,1} = \frac{1}{2} Q_{.\lambda\mu 1}, \quad R_{1\lambda\mu 1,\gamma} = \frac{1}{2} Q_{.\lambda\mu\gamma},$$

$$C_{1\lambda\mu 1} = \frac{1}{2}\left[ Q_{.\lambda\mu} - \frac{1}{n-2} k_{\lambda\mu} (k^{\alpha\beta} Q_{.\alpha\beta}) \right], \tag{43}$$

$$C_{1\lambda\mu 1,1} = \frac{1}{2}\left[ Q_{.\lambda\mu 1} - \frac{1}{n-2} k_{\lambda\mu} (k^{\alpha\beta} Q_{.\alpha\beta 1}) \right],$$

$$C_{1\lambda\mu 1,\gamma} = \frac{1}{2}\left[ Q_{.\lambda\mu\gamma} - \frac{1}{n-2} k_{\lambda\mu} (k^{\alpha\beta} Q_{.\alpha\beta\gamma}) \right],$$

and that all other components are identically zero.

Since $M$ is conformally recurrent and conditions (5) and (2) hold, $b_{i,j} = b_{j,i}$ on some neighbourhood of $x$. The recurrence vector $b_j$ is therefore a gradient.

Hence, in view of (43),

$$\left[Q_{.\lambda\mu} - \frac{1}{n-2} k_{\lambda\mu}(k^{\alpha\beta}Q_{.\alpha\beta})\right]_{.j} = C_{.j}\left[Q_{.\lambda\mu} - \frac{1}{n-2} k_{\lambda\mu}(k^{\alpha\beta}Q_{.\alpha\beta})\right]$$

for some function $C$.

Thus

$$Q_{.\lambda\mu} = Ak_{\lambda\mu} + Bc_{\lambda\mu},$$

where $[c_{\lambda\mu}]$ is a symmetric and non-zero matrix satisfying $k^{\alpha\beta}c_{\alpha\beta} = 0$.

Moreover, because of (40) and (43), we have

$$R_{11} = \frac{n-2}{2}A, \quad R_{11,j} = \frac{n-2}{2}A_{.j}, \quad C_{1\lambda\mu 1} = \frac{1}{2}Bc_{\lambda\mu}, \quad C_{1\lambda\mu 1,j} = \frac{1}{2}B_{.j}c_{\lambda\mu},$$

(44)

$$R_{1\lambda\mu 1} = \frac{1}{2}(Ak_{\lambda\mu} + Bc_{\lambda\mu}), \quad R_{1\lambda\mu 1,j} = \frac{1}{2}(A_{.j}k_{\lambda\mu} + B_{.j}c_{\lambda\mu}).$$

Since conditions (28), (32), and $C_{hijk} \neq 0$ are satisfied by assumption, $A$ and $B$, as one can easily verify, satisfy (41).

Suppose now that the conditions of (ii) hold. Then, as follows easily from (44) and (1), $(R^n, g)$ is a conformally recurrent Ricci-recurrent manifold. Since $B \neq \text{const}$ and $AB_{.j} - BA_{.j}$ does not identically vanish, $(R^n, g)$ is neither conformally symmetric nor recurrent. Moreover, because of $C_{1\lambda\mu 1,ij} - C_{1\lambda\mu 1,ji} = 0$, condition (5) is satisfied. This completes the proof.

Remark 3. If $Q$ in (39) is of the form $Q = (Ak_{\lambda\mu} + Bc_{\lambda\mu})x^\lambda x^\mu$, where $A$ and $B$ are functions of $x^1$ only, satisfying assumptions of (ii) and $A \neq 0 \neq B$, then $(R^n, g)$ is an essentially conformally recurrent Ricci-recurrent manifold such that the recurrence vectors of $R_{ij}$ as well as of $C_{hijk}$ are both null at each point of $R^n$.

Now we shall prove the existence of essentially conformally recurrent Ricci-recurrent manifolds whose recurrence vectors are non-null everywhere.

Suppose that $R^n$ is endowed with the metric (39), where

$$[k_{\lambda\mu}] = \begin{bmatrix} e_2 & & 0 \\ & \ddots & \\ 0 & & e_{n-1} \end{bmatrix}, \quad |e_\lambda| = 1,$$

and

$$Q = \frac{1}{2}\sum_{i=2}^{n-1} e_i(x^i)^2 + (n-2)e_{n-1}\exp(x^{n-1}).$$

Then, as one can easily verify, we have

$$Q_{.\lambda\mu} = \big(1+\exp(x^{n-1})\big)k_{\lambda\mu}+c_{\lambda\mu}\exp(x^{n-1}),$$

where

$$[c_{\lambda\mu}] = \begin{bmatrix} -e_2 & & & & \\ & -e_3 & & 0 & \\ & & \ddots & & \\ & 0 & & -e_{n-2} & \\ & & & & (n-3)e_{n-1} \end{bmatrix}.$$

Since we have $k^{\alpha\beta}c_{\alpha\beta} = 0$ and the functions $A = 1+\exp(x^{n-1})$ and $B = \exp(x^{n-1})$ satisfy the condition $AB_{.j}-BA_{.j} \neq 0$, $(R^n, g)$ is an essentially conformally recurrent Ricci-recurrent manifold.

Taking now into account (44), we obtain

$$a_j = \partial_j \log\big(1+\exp(x^{n-1})\big) \quad \text{and} \quad b_j = \delta_j^{n-1},$$

where $a_j$ and $b_j$ denote the recurrence vectors of $R_{ij}$ and $C_{hijk}$, respectively. Moreover, as one can easily verify, $a^r a_r \neq 0 \neq b^r b_r$ everywhere on $R^n$.

Remark 4. The metric (4) can be obtained from (40) as follows: Let

$$A = \frac{2}{n-2}\bar{A}+2 \quad \text{and} \quad Bc_{\lambda\mu} = 2\bar{A}\left(p_{\lambda\mu}-\frac{1}{n-2}k_{\lambda\mu}\right),$$

where $p_{\lambda\mu} = p_{\mu\lambda} = \text{const}$ and $\bar{A}$ is a non-constant function of $x^1$ only. Then $k^{\alpha\beta}p_{\alpha\beta} = 1$ and $Q_{.\lambda\mu} = 2(\bar{A}p_{\lambda\mu}+k_{\lambda\mu})$. Hence

$$Q = (\bar{A}p_{\lambda\mu}+k_{\lambda\mu})x^\lambda x^\mu+\varrho_a x^a+\eta,$$

where $\varrho_a$, $\eta$ are functions of $x^1$ only. Taking $\varrho_a = \eta = 0$, we obtain (4).

Remark 5. The subset of points satisfying (28), (32), and $C_{hijk} \neq 0$ is dense if $(M, g)$ is an analytic non-recurrent Ricci-recurrent manifold (with non-parallel Ricci tensor).

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