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CLASSICAL SOLUTIONS OF THE TWO-PHASE STEFAN-TYPE PROBLEM

BY

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1. Introduction. In this paper we shall be concerned with the following two-sided Stefan-type problem for the heat equation:

Given T, $0 < T < \infty$, a function $s(t)$, $0 < s(t) < 1$, $0 \leqslant t < T$, defines two regions Ω_T and Ω'_T:

$$(1.1)\qquad \begin{aligned} \Omega_T &= \{(x, t) \in \boldsymbol{R} \times (0, T):\ 0 < x < s(t)\}, \\ \Omega'_T &= \{(x, t) \in \boldsymbol{R} \times (0, T):\ s(t) < x < 1\}. \end{aligned}$$

Now we look for a function $s(t)$, $0 < s(t) < 1$, and two other functions $u(x, t)$ and $v(x, t)$ satisfying:

(a) the equations

$$(1.2)\qquad u_t = a^2 u_{xx} \quad \text{in } \Omega_T,$$

$$(1.3)\qquad v_t = b^2 v_{xx} \quad \text{in } \Omega'_T$$

with a and b some positive constants;

(b) the initial conditions

$$(1.4)\qquad s(0) = d,\ 0 < d < 1, \quad \dot{s}(0) = c,\ c \in \boldsymbol{R},$$

$$(1.5)\qquad u(x, 0) = p(x),\ 0 \leqslant x \leqslant d, \quad v(x, 0) = q(x),\ d \leqslant x \leqslant 1,$$

p and q given continuous functions;

(c) the boundary conditions for $x = 0$ and $x = 1$

$$(1.6)\qquad u(0, t) = \varphi(t), \quad v(1, t) = \psi(t), \quad 0 \leqslant t < T,$$

φ and ψ given continuous functions satisfying compatibility conditions $\varphi(0) = p(0)$ and $\psi(0) = q(1)$;

(d) the boundary conditions on the free boundary $x = s(t)$, $0 < t < T$,

$$(1.7)\qquad \ddot{s}(t) = H(s(t), \dot{s}(t), u(t), v(t), t),$$

$$u_x(s(t), t) = f(s(t), \dot{s}(t), u(t), v(t), t),$$
(1.8)
$$v_x(s(t), t) = g(s(t), \dot{s}(t), u(t), v(t), t).$$

In these equations we have used the following notation:

(1.9) $$u(t) = u(s(t), t), \qquad v(t) = v(s(t), t),$$

f, g and H are continuous functions defined in some neighbourhood of the set A:

(1.10) $$A = [0, 1] \times \boldsymbol{R} \times \{|u| \leqslant 4\|\varphi\|+1\} \times \{|v| \leqslant 4\|\psi\|+1\} \times [0, T],$$

where

$$\|\varphi\| = \sup_{0 \leqslant t \leqslant T} |\varphi(t)|,$$

satisfying the Lipschitz condition with respect to all but the last argument. Due to (1.7) our problem is a free boundary problem with (1.7) describing the dynamics of a material wall separating two different media.

Physically, functions f and g describe influence of the flux of heat flow through a point $x = s(t)$. The process itself changes the position of the point $x = s(t)$.

We say that a triple (u, s, v) forms a *classical solution* of our problem if

$$u_{xx}, u_t \in C(\Omega_T), \qquad v_{xx}, v_t \in C(\Omega'_T),$$
$$u_x \in C(\bar{\Omega}_T \cap \boldsymbol{R} \times (0, T)), \qquad v_x \in C(\bar{\Omega}'_T \cap \boldsymbol{R} \times (0, T)),$$
$$u \in C(\bar{\Omega}_T), \qquad v \in C(\bar{\Omega}'_T),$$
$$\ddot{s} \in C([0, T])$$

and u, s and v satisfy (1.2)–(1.8).

The main result of this paper is the following

THEOREM 1. *Let φ, ψ, p, q and H be as above. If $T > 0$ is sufficiently small, then there exists a unique solution (u, s, v) of the system* (1.2)–(1.8).

G. Lamé, B. P. Clapeyron (1831) and J. Stefan (1889) were probably the first who considered free boundary problems. Since that time many authors have studied them by using various methods (see [1]–[3], [8] and [10]). For the one-sided Stefan-type problem, the existence of a unique solution, local and global in time, has been proved by Tabisz [9]. We acknowledge useful discussion with A. Krzywicki and K. Tabisz during the preparation of this paper.

2. Integral equation. It is well known that if a pair (u, v) forms a classical solution of (1.2)–(1.6) for $0 < t < T$, then u, v satisfy the integral equation (cf.

[5], Sections 60 and 61)

$$(2.1)\quad u(x,t)=a^2\int_0^t G_\xi(x,t,0,\tau)\varphi(\tau)\,d\tau+\int_0^d G(x,t,\xi,0)p(\xi)\,d\xi$$

$$+\int_0^t [G(x,t,s(\tau),\tau)\dot s(\tau)-a^2G_\xi(x,t,s(\tau),\tau)]u(s(\tau),\tau)\,d\tau$$

$$+a^2\int_0^t G(x,t,s(\tau),\tau)f(s(\tau),\dot s(\tau),u(\tau),v(\tau),\tau)\,d\tau,\qquad (x,t)\in\Omega_T,$$

$$(2.2)\quad v(x,t)=-b^2\int_0^t G'_\xi(x,t,1,\tau)\psi(\tau)\,d\tau-\int_d^1 G'(x,t,\xi,0)q(\xi)\,d\xi$$

$$-\int_0^t [G'(x,t,s(\tau),\tau)\dot s(\tau)-b^2G'_\xi(x,t,s(\tau),\tau)]v(s(\tau),\tau)\,d\tau$$

$$-b^2\int_0^t G'(x,t,s(\tau),\tau)g(s(\tau),\dot s(\tau),u(\tau),v(\tau),\tau)\,d\tau,\qquad (x,t)\in\Omega'_T,$$

G and G' are the Green functions for the heat equation in the half-planes $x>0$ and $x<1$, respectively:

(2.3)
$$G(x,t,\xi,\tau)=\frac{1}{2a(\pi(t-\tau))^{1/2}}\left[\exp\left(-\frac{(x-\xi)^2}{4a^2(t-\tau)}\right)-\exp\left(-\frac{(x+\xi)^2}{4a^2(t-\tau)}\right)\right],$$

(2.4)
$$G'(x,t,\xi,\tau)=\frac{1}{2b(\pi(t-\tau))^{1/2}}\left[\exp\left(-\frac{(x-\xi)^2}{4b^2(t-\tau)}\right)-\exp\left(-\frac{(x+\xi-2)^2}{4b^2(t-\tau)}\right)\right].$$

THEOREM 2. *If a triple (u, s, v) forms a classical solution of the system* (1.2)–(1.8) *for some $T>0$, then $u(t)$ and $v(t)$ (cf.* (1.9) *for definition) are solutions of the system*

$$(2.5)\quad u(t)=2a^2\int_0^t G_\xi(s(t),t,0,\tau)\varphi(\tau)\,d\tau$$

$$+2\int_0^d G(s(t),t,\xi,0)p(\xi)\,d\xi$$

$$+2\int_0^t [G(s(t),t,s(\tau),\tau)\dot s(\tau)-a^2G_\xi(s(t),t,s(\tau),\tau)]u(\tau)\,d\tau$$

$$+2a^2\int_0^t G(s(t),t,s(\tau),\tau)f(s(\tau),\dot s(\tau),u(\tau),v(\tau),\tau)\,d\tau,$$

$$(2.6)\quad v(t) = -2b^2\int_0^t G'_\xi(s(t), t, 1, \tau)\psi(\tau)d\tau - 2\int_d^1 G'(s(t), t, \xi, 1)q(\xi)d\xi$$

$$-2\int_0^t [G'(s(t), t, s(\tau), \tau)\dot{s}(\tau) - b^2 G'_\xi(s(t), t, s(\tau), \tau)]\, v(\tau)d\tau$$

$$-2b^2\int_0^t G'(s(t), t, s(\tau), \tau)g(s(\tau), \dot{s}(\tau), u(\tau), v(\tau), \tau)d\tau,$$

where $s(t)$ is a solution of ordinary differential equation (1.7).

Conversely, if for some $T > 0$, (u, s, v) forms a solution of the system (1.7), (2.5), (2.6) *and, moreover,* $0 < s(t) < 1$, $0 < t < T$, *then $u(x, t)$ and $v(x, t)$ given by* (2.1) *and* (2.2) *together with $s(t)$ form a classical solution of the system* (1.2)–(1.8).

Proof. Using (2.1), (2.2) and (1.7)–(1.8) we obtain (2.5) and (2.6) by taking $x \to s(t) \pm 0$ and making use of the known properties of heat potentials (cf. [5]).

Conversely, let $\tilde{u}$, $\tilde{v}$ and s be the solutions of the system (1.7), (2.5), (2.6). It is well known that $u(x, t)$ and $v(x, t)$ defined by (2.1) and (2.2) satisfy the heat equations $u_t = a^2 u_{xx}$ in Ω_T and $v_t = b^2 v_{xx}$ in Ω'_T respectively, regularity conditions $u_t, u_{xx} \in C(\Omega_T)$ and $v_t, v_{xx} \in C(\Omega'_T)$, $u \in C(\bar{\Omega}_T)$, $v \in C(\bar{\Omega}'_T)$, and boundary conditions $u(x, 0) = p(x)$, $v(x, 0) = q(x)$, $u(0, t) = \varphi(t)$, $v(1, t) = \psi(t)$, with compatibility conditions $\varphi(0) = p(0)$ and $\psi(0) = q(1)$ (cf. [3], Section 5).

Now passing to the limit $x \to s(t) \pm 0$ in (2.1) and (2.2) we obtain

$$u(s(t), t) = \tilde{u}(t), \qquad v(s(t), t) = \tilde{v}(t),$$

and hence (1.7) on the free boundary $x = s(t)$. To show that

$$u_x \in C(\Omega_T \cap \boldsymbol{R} \times (0, T)) \quad \text{and} \quad v_x \in C(\Omega'_T \cap \boldsymbol{R} \times (0, T))$$

we can use the argument of Tabisz [9]. We consider the auxiliary problem:

$$\bar{u}_t = a^2 \bar{u}_{xx} \quad \text{and} \quad \bar{v}_t = b^2 \bar{v}_{xx}$$

satisfying

$$\bar{u}(0, t) = \varphi(t), \qquad \bar{u}(x, 0) = p(x),$$

$$\bar{u}_x(s(t), t) = f(s(t), \dot{s}(t), u(t), v(t), t)$$

for $0 < t < T$ and $0 \leqslant x \leqslant d$; and

$$\bar{v}(1, t) = \psi(t), \qquad \bar{v}(x, 0) = q(x),$$

$$\bar{v}_x(s(t), t) = g(s(t), \dot{s}(t), u(t), v(t), t)$$

for $0 < t < T$ and $d \leqslant x \leqslant 1$. The method of heat potentials guarantees the existence and uniqueness of $\bar{u}$ and $\bar{v}$ and continuity of $\bar{u}_x$ and $\bar{v}_x$ up to the boundary. If we express $\bar{u}$ and $\bar{v}$ in the forms (2.1) and (2.2) and compare with the original formulas (2.1) and (2.2), passing to the limit $x \to s(t) \pm 0$ we get

$$\bar{u}(t) - u(t) = \int_0^t (G\dot{s} - a^2 G_\xi)(\bar{u} - u)\, d\tau,$$

$$\bar{v}(t) - v(t) = -\int_0^t (G'\dot{s} - b^2 G'_\xi)(\bar{v} - v)\, d\tau.$$

Hence, due to the integrability with respect to τ of kernels $G\dot{s} - a^2 G_\xi$ and $G'\dot{s} - b^2 G'_\xi$ we obtain $\bar{u} = u$, $\bar{v} = v$ and, in consequence, $u_x = f$, $v_x = g$ on the free boundary.

3. Existence and uniqueness for small time. Due to Theorem 2 it suffices to prove the existence and uniqueness of a solution for the system (1.7), (2.5), (2.6). This system can be solved by applying the Banach contraction theorem. This method has appeared first in [6], and explicitly in Friedman's memoir [4].

THEOREM 3. *For sufficiently small $T > 0$ there exists a unique solution of the system* (1.7), (2.5), (2.6).

Proof. Let $C[0, T]$ be the Banach space of continuous functions defined on $[0, T]$ with supremum norm. Let B_M^T denote the closed ball centred at $(0, 0)$ and of radius $M > 0$ in $C[0, T] \oplus C[0, T]$ (with the norm $\|(x, y)\| = \|x\| + \|y\|$). We define a transformation

$$\Phi \colon B_M^T \to C[0, T] \oplus C[0, T]$$

putting $\Phi(u, v) = (\bar{u}, \bar{v})$, where $\bar{u}$ and $\bar{v}$ are the right-hand sides of (2.5) and (2.6), respectively, with s given by (1.7).

LEMMA 1. *For every $M > 0$ and $T > 0$ there exists T_1, $0 < T_1 \leqslant T$, such that, for any $(u, v) \in B_M^T$, the equation* (1.7) *with the initial conditions* (1.4) *has a unique solution s defined on $[0, T_1]$ such that*

$$|s(t) - d| \leqslant 2^{-1} \min\{d, 1-d\} \equiv \alpha \quad \textit{for } t \in [0, T_1], \tag{3.1}$$

$$|\dot{s}(t)| \leqslant w \quad \textit{for some } w \geqslant 0 \textit{ and } t \in [0, T_1]. \tag{3.2}$$

Furthermore, if $s_i(t)$, $i = 1, 2$, are solutions of (1.7) *corresponding to $(u_i, v_i) \in B_M^T$, respectively, and $s_i(0) = d$, $\dot{s}_i(0) = c$, then there exists a constant $N \geqslant 0$ such that*

$$\begin{aligned} |s_1(t) - s_2(t)| &\leqslant Nt\{\|u_1 - u_2\| + \|v_1 - v_2\|\}, \quad t \in [0, T_1], \\ |\dot{s}_1(t) - \dot{s}_2(t)| &\leqslant Nt\{\|u_1 - u_2\| + \|v_1 - v_2\|\}, \quad t \in [0, T_1]. \end{aligned} \tag{3.3}$$

The proof of this lemma is similar to that of the theorem on continuous dependence on the parameter of solutions of ordinary differential equations (cf. [7]).

LEMMA 2. *There exist constants $M>0$ and T_2, $0<T_2\leqslant T_1$, such that*

$$\Phi(B_M^T)\subset B_M^T \quad \textit{for all } T\ (0<T\leqslant T_2).$$

Proof. Without restriction of generality of our considerations we can put $p=q=0$ (cf. [9]). Let us write (2.5) in the form

$$\bar{u}=I_1+\ldots+I_4$$

with I_i $(i=1,\ldots,4)$ denoting successive integrals appearing in (2.5). Now we shall estimate I_i $(i=1,\ldots,4)$ similarly as in [4]. Applying the substitution

$$\beta=\frac{s(t)}{2a(t-\tau)^{1/2}}$$

and using the fact that

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx=2(\pi)^{1/2}$$

we obtain

$$|I_1|\leqslant 2a^2\int_0^t |G_\xi|\,|\varphi|\,d\tau\leqslant 4\,\|\varphi\|$$

and

$$|I_2|\leqslant 2\int_0^t |G|\,|\dot{s}|\,|u|\,d\tau\leqslant C\,\|u\|\,t^{1/2}$$

for some constant $C\geqslant 0$. By (3.2) and (3.1) we have

$$\begin{aligned}
|I_3| &\leqslant 2a^2\int_0^t |G_\xi|\,|u|\,d\tau\\
&\leqslant \|u\|\int_0^t \frac{1}{2a\big(\pi(t-\tau)\big)^{1/2}}\left|\frac{s(t)-s(\tau)}{t-\tau}\right|\exp\left(\frac{\big(s(t)-s(\tau)\big)^2}{4a^2(t-\tau)}\right)d\tau\\
&\quad+\|u\|\int_0^t \frac{1}{2a\big(\pi(t-\tau)\big)^{1/2}}\left|\frac{s(t)+s(\tau)}{t-\tau}\right|\exp\left(-\frac{\big(s(t)+s(\tau)\big)^2}{4a^2(t-\tau)}\right)d\tau\\
&\leqslant C\,\|u\|\,t^{1/2}.
\end{aligned}$$

Of course, for $\big(s(\tau),\dot{s}(\tau),u(\tau),v(\tau),\tau\big)\in A$ (cf. (1.10)) by (3.2) and the Lipschitz condition for f we obtain

$$\big|f\big(s(\tau),\dot{s}(\tau),u(\tau),v(\tau),\tau\big)\big|\leqslant C\,(\|u\|+\|v\|+1),$$

and hence

$$|I_4|\leqslant 2a^2\int_0^t |G|\,|f|\,d\tau\leqslant C\,(\|u\|+\|v\|+1)\,t^{1/2}.$$

Hence, for $M = 4\|\varphi\|+1$ and $T_2^* = \min\{T_1, [(4M+1)C]^{-2}\}$ we obtain $\|u\| \leqslant M$, which implies $\|\bar{u}\| \leqslant M$ in $[0, T_2^*]$.

Writing similarly $\bar{v} = J_1 + \ldots + J_4$ for $\bar{v}$ given by (2.6) we have, by Lemma 1,

$$|J_1| \leqslant 2b^2 \int_0^t |G'_\xi|\,|\psi|\,d\tau$$

$$\leqslant 2b^2 \frac{\|\psi\|}{\pi^{1/2}} \left[\int_0^t \frac{s(t)}{2b(t-\tau)^{3/2}} \exp\left(-\frac{(s(t))^2}{4b^2(t-\tau)}\right) d\tau + \int_0^t \frac{2-s(t)}{2b(t-\tau)^{3/2}} \exp\left(-\frac{(2-s(t))^2}{4b^2(t-\tau)}\right) d\tau \right].$$

Substituting

$$\beta = \frac{s(t)}{2b(t-\tau)^{1/2}} \quad \text{and} \quad \gamma = \frac{2-s(t)}{2b(t-\tau)^{1/2}},$$

respectively, we obtain $|J_4| \leqslant 4\|\psi\|$. Similarly as before we get

$$|J_2| \leqslant 2\int_0^t |G'|\,|\dot{s}|\,|v|\,d\tau \leqslant C\|v\|\,t^{1/2}, \qquad 0 \leqslant t \leqslant T_1.$$

By (3.1) and $|s(t)+s(\tau)-2| > d$ for $0 \leqslant t \leqslant T_1$ we have

$$|J_3| \leqslant 2b^2 \int_0^t |G'_\xi|\,|v|\,d\tau \leqslant C\|v\|\,t^{1/2}, \qquad 0 \leqslant t \leqslant T_1.$$

Also, by the argument similar to that used in estimating I_4, we obtain

$$|J_4| \leqslant 2b^2 \int_0^t |G'|\,|g|\,d\tau \leqslant C(\|u\|+\|v\|+1)\,t^{1/2}, \qquad 0 \leqslant t \leqslant T_1.$$

Putting

$$M = \max\{4\|\varphi\|+1,\ 4\|\psi\|+1\} \quad \text{and} \quad T_2 = \min\{T_2^*, [(4M+1)C]^{-2}\}$$

we conclude that $\|v\| \leqslant M$ implies $\|\bar{v}\| \leqslant M$ in $[0, T_2]$, and hence

$$\Phi(B_M^T) \subset B_M^T \quad \text{for any } T \in [0, T_2).$$

LEMMA 3. *There exists a constant T_3, $0 < T_3 \leqslant T_2$, such that the mapping Φ: $B_M^T \to B_M^T$, $0 < T \leqslant T_3$ (with $M > 0$ provided by Lemma 2) is a contraction.*

Proof. Let $0 < T \leqslant T_2$ and $(u_1, v_1), (u_2, v_2) \in B_M^T$. To estimate

$$\Phi(u_1, v_1) - \Phi(u_2, v_2)$$

we write

$$\bar{u}_1 - \bar{u}_2 = \bar{I}_1 + \ldots + \bar{I}_4 \quad \text{and} \quad \bar{v}_1 - \bar{v}_2 = \bar{J}_1 + \ldots + \bar{J}_4, \tag{3.4}$$

where $\bar{I}_i$, $\bar{J}_i$ ($i = 1, \ldots, 4$) denote the differences of successive integrals appearing on the right-hand side in (2.5), (2.6). Further we denote by

$$G^i = G(s_i(t), \ldots), \quad f^i = f(s_i(t), \ldots), \quad \ldots$$

the functions $G, f, \ldots$ appearing in (2.5) and (2.6) which correspond to (u_i, v_i), $i = 1, 2$, respectively.

By Lemma 1, $s_i(t) \geqslant d-\alpha > 0$ for $0 \leqslant t \leqslant T_2$, so by the Mean Value Theorem we obtain

$$|\bar{I}_1| \leqslant 2a^2 \int_0^t |G_\xi^1 - G_\xi^2| \, |\varphi| \, d\tau \leqslant C \|s_1 - s_2\| t^{1/2}, \quad 0 \leqslant t \leqslant T_2.$$

We now apply to every summand in $\bar{I}_2$ the algebraic identity

$$\bar{A}\bar{B}\bar{C} - ABC = (\bar{A}-A)\bar{B}\bar{C} + (\bar{B}-B)A\bar{C} + (\bar{C}-C)AB$$

and proceed similarly as in [4]:

$$|\bar{I}_2| \leqslant 2\int_0^t |G^1 \dot{s}_1 u_1 - G^2 \dot{s}_2 u_2| \, d\tau$$

$$\leqslant 2\int_0^t |G^2| \, |\dot{s}_2| \, |u_1 - u_2| \, d\tau + 2\int_0^t |G^2| \, |\dot{s}_1 - \dot{s}_2| \, |u_1| \, d\tau$$

$$+ 2\int_0^t |G^1 - G^2| \, |\dot{s}_1| \, |u_1| \, d\tau \equiv I_{21} + I_{22} + I_{23}.$$

By Lemma 2 we obtain

$$I_{21} \leqslant C \|u_1 - u_2\| t^{1/2},$$

$$I_{22} \leqslant C \|\dot{s}_1 - \dot{s}_2\| t^{1/2} \leqslant C(\|u_1 - u_2\| + \|v_1 - v_2\|) t^{1/2}, \quad 0 \leqslant t \leqslant T_2.$$

We have also $I_{23} \leqslant I_{231} + I_{232}$, where

$$I_{231} = 2Mw \int_0^t \frac{1}{2a(\pi(t-\tau))^{1/2}} \times \left| \exp\left(-\frac{(s_1(t) - s_1(\tau))^2}{4a^2(t-\tau)}\right) - \exp\left(-\frac{(s_2(t) - s_2(\tau))^2}{4a^2(t-\tau)}\right) \right| d\tau,$$

$$I_{232} = 2Mw \int_0^t \frac{1}{2a(\pi(t-\tau))^{1/2}} \times \left| \exp\left(-\frac{(s_1(t) + s_1(\tau))^2}{4a^2(t-\tau)}\right) - \exp\left(-\frac{(s_2(t) + s_2(\tau))^2}{4a^2(t-\tau)}\right) \right| d\tau.$$

Using the inequality $|e^{-x} - e^{-y}| \leqslant |x-y|$ for $x, y \geqslant 0$, we have

$$I_{231} \leqslant Ct^{1/2}(\|u_1 - u_2\| + \|v_1 - v_2\|), \quad 0 \leqslant t \leqslant T_2.$$

Likewise, by (3.1) and the Mean Value Theorem, we obtain

$$I_{232} \leqslant Ct^{1/2}(\|u_1-u_2\|+\|v_1-v_2\|), \quad 0 \leqslant t \leqslant T_2.$$

Similarly, we have

$$\begin{aligned}|\bar{I}_3| &\leqslant 2a^2 \int_0^t |G_\xi^1 u_1 - G_\xi^2 u_2|\,d\tau \\ &\leqslant 2a^2 \int_0^t |u_1-u_2|\,|G_\xi^1|\,d\tau + 2a^2 \int_0^t |G_\xi^1 - G_\xi^2|\,|u_2|\,d\tau \\ &\leqslant 2a^2(\|u_1-u_2\|\,\|\dot{s}\| + M\|\dot{s}_1-\dot{s}_2\|) + C(\|u_1-u_2\|+\|v_1-v_2\|)t^{1/2},\end{aligned}$$

whence

$$|\bar{I}_3| \leqslant C(\|u_1-u_2\|+\|v_1-v_2\|)t^{1/2}, \quad 0 \leqslant t \leqslant T_2.$$

To estimate $\bar{I}_4$, we write

$$\begin{aligned}|\bar{I}_4| &\leqslant 2a^2 \int_0^t |G^1 f^1 - G^2 f^2|\,d\tau \\ &\leqslant 2a^2\Big(\int_0^t |G^1-G^2|\,|f^2|\,d\tau + \int_0^t |G^1|\,|f^1-f^2|\,d\tau\Big).\end{aligned}$$

Then, by Lemma 1 and the Lipschitz condition for f, we have

$$|\bar{I}_4| \leqslant C(\|u_1-u_2\|+\|v_1-v_2\|)t^{1/2}, \quad 0 \leqslant t \leqslant T_2.$$

Hence, by (3.4),

(3.5)
$$\|\bar{u}_1-\bar{u}_2\| \leqslant C(\|u_1-u_2\|+\|v_1-v_2\|)t^{1/2} \quad \text{for some } C \geqslant 0,\ 0 \leqslant t \leqslant T_2.$$

In an analogous way we can establish an estimate of $\bar{v}_1-\bar{v}_2$:

(3.6)
$$\|\bar{v}_1-\bar{v}_2\| \leqslant C(\|u_1-u_2\|+\|v_1-v_2\|)t^{1/2}, \quad 0 \leqslant t \leqslant T_2.$$

The inequality (3.6) together with (3.5) implies the existence of $T_3 > 0$ such that the map Φ is a contraction on B_M^T, $0 < T \leqslant T_3$. This completes the proof of Lemma 3 and, consequently, of Theorem 1.

REFERENCES

[1] J. R. Cannon, D. B. Henry and D. B. Kotlow, *Classical solution of the one-dimensional, two-phase Stefan problem*, Ann. Mat. Pura Appl. 4 (107) (1975), pp. 311–341.

[2] A. Fasano and M. Primicerio, *New result on some classical parabolic free boundary problems*, Quart. Appl. Math. (to appear).

[3] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N. J., 1964.

[4] – *Free boundary problem for parabolic equations, I. Melting of solids*, J. Math. Mech. 8 (1959), pp. 499–518.

[5] M. Krzyżański, *Partial Differential Equations of the Second Order*, PWN–Polish Scientific Publishers, Warszawa 1971.

[6] L. I. Rubinstein, *The Stefan Problem*, Zvaigznye, Riga 1967.

[7] L. Schwartz, *Analyse Mathématique*, Hermann, Paris 1967.

[8] K. Tabisz, *On a kind of the Stefan problem*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 27 (1979), pp. 847–852.

[9] – *Local and global solution of the Stefan-type problem*, J. Math. Anal. Appl. 82 (1981), pp. 306–316.

[10] D. A. Tarzia, *Sur le problème de Stefan à deux phases*, C. R. Acad. Sci. Paris Sér. A–B 288–A (1979), pp. 941–944.

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