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A PROPOSAL CONCERNING THE FORMULATION OF THE INFINITISTIC AXIOM IN THE THEORY OF LOGICAL PROBABILITY

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1. Let $\boldsymbol{L}$ be the set of closed sentences of some language. The elements of $\boldsymbol{L}$ are of finite length and the set $\boldsymbol{L}$ is closed with respect to the usual syntactic operations of sentential logic, i.e. the sentences $\sim\alpha$, $\alpha\wedge\beta$, $\alpha\vee\beta$, $\alpha\rightarrow\beta$, $\alpha\leftrightarrow\beta$ are in $\boldsymbol{L}$ whenever the sentences α,β are in $\boldsymbol{L}$. The set of all tautologies of sentential logic formulated in $\boldsymbol{L}$ is denoted by the letter T.

Let $\boldsymbol{C}$ be an operation of logical consequence defined for subsets of $\boldsymbol{L}$. It means that for every $X, Y\subset\boldsymbol{L}$:

1° $X\subset\boldsymbol{C}(X)\subset\boldsymbol{L}$;

2° if $X\subset\boldsymbol{C}(Y)$, then $\boldsymbol{C}(X)\subset\boldsymbol{C}(Y)$.

The set X is called *consistent* if $\boldsymbol{C}(X)\neq\boldsymbol{L}$. The symbol $X^{(\alpha)}$ denotes the set formed by the elements of X and the sentence α. We will write $\boldsymbol{C}_X(\alpha)$ instead of $\boldsymbol{C}(X^{(\alpha)})$.

It is assumed that the operation $\boldsymbol{C}$ fulfils the following conditions:

3° $T\subset\boldsymbol{C}(\emptyset)\neq\boldsymbol{L}$;

4° $\beta\in\boldsymbol{C}_X(\alpha)$ if and only if $(\alpha\rightarrow\beta)\in\boldsymbol{C}(X)$.

Consequently,

5° $\boldsymbol{C}_X(\alpha)=\boldsymbol{L}$ if and only if $(\sim\alpha)\in\boldsymbol{C}(V)$.

2. Let $\boldsymbol{p}(\alpha, A)$ be a function with values in the set of real numbers and defined for every sentence α in $\boldsymbol{L}$ and every subset A of $\boldsymbol{L}$. If the function $\boldsymbol{p}(\alpha, A)$ satisfies the axioms given below, then we call it *logical probability function based on the logic* $\langle\boldsymbol{L},\boldsymbol{C}\rangle$. Instead of the notations $\boldsymbol{p}(\alpha, A)$ and $\boldsymbol{p}(\alpha, A^{(\beta)})$ we will write $\boldsymbol{p}_A(\alpha)$ and $\boldsymbol{p}_A(\alpha\,|\,\beta)$, respectively.

The following axiom system (I, II, III, IV, V) is a modification of that of Mazurkiewicz [2]:

I. $0\leqslant\boldsymbol{p}_A(\alpha)$.

II. $\boldsymbol{p}_A(\alpha)=1$ if and only if $\alpha\in\boldsymbol{C}(A)$.

III. If the set A is consistent and the set $A^{(\alpha\wedge\beta)}$ is inconsistent, then $\boldsymbol{p}_A(\alpha\vee\beta)=\boldsymbol{p}_A(\alpha)+\boldsymbol{p}_A(\beta)$.

IV. $\boldsymbol{p}_A(\alpha \wedge \beta) = \boldsymbol{p}_A(\alpha|\beta)\cdot \boldsymbol{p}_A(\beta)$.

V. If $\boldsymbol{C}(A) = \boldsymbol{C}(B)$, then $\boldsymbol{p}_A(\alpha) = \boldsymbol{p}_B(\alpha)$.

The axioms I, II and III entail the following theorems:

(1) $\boldsymbol{p}_A(\alpha \leftrightarrow \beta) = 1$ *if and only if* $\boldsymbol{C}_A(\alpha) = \boldsymbol{C}_A(\beta)$.

(2) *If* $\boldsymbol{C}(A) = \boldsymbol{L}$, *then* $\boldsymbol{p}_A(\alpha) = 1$ *for every* $\alpha \in \boldsymbol{L}$.

(3) *If* $\boldsymbol{C}(A) \neq \boldsymbol{L}$, *then* $\boldsymbol{p}_A(\alpha) = 0$ *if and only if* $(\sim \alpha) \in \boldsymbol{C}(A)$.

3. Let us recall a construction of Kolmogorov [1]. One may define the ***p**-distance* over A of the sentences α, β as follows:

$$D_A^{\boldsymbol{p}}(\alpha, \beta) = 1 - \boldsymbol{p}_A(\alpha \leftrightarrow \beta).$$

Clearly,

(4) $0 \leqslant D_A^{\boldsymbol{p}}(\alpha, \beta) \leqslant 1$.

(5) $D_A^{\boldsymbol{p}}(\alpha, \beta) = 0$ *if and only if* $\boldsymbol{C}_A(\alpha) = \boldsymbol{C}_A(\beta)$.

(6) $D_A^{\boldsymbol{p}}(\alpha, \beta) = D_A^{\boldsymbol{p}}(\beta, \alpha)$.

(7) $D_A^{\boldsymbol{p}}(\alpha, \beta) \leqslant D_A^{\boldsymbol{p}}(\alpha, \gamma) + D_A^{\boldsymbol{p}}(\beta, \gamma)$.

The sequence $\{\alpha_n\}_{n=1,2,\ldots}$ of sentences is called ***p**-convergent over* A *to the sentence* α if and only if $\lim\limits_{n=\infty} D_A^{p}(\alpha_n, \alpha) = 0$. It is easy to show the following theorem (of kolmogorov):

(8) *Suppose that the set A is consistent and the sequence* $\{\alpha_1 \vee \ldots \vee \alpha_n\}_{n=1,2,\ldots}$ *is **p**-convergent over A to the sentence α. If the sets* $A^{(\alpha_i \wedge \alpha_j)}$ *are inconsistent for* $i \neq j$, *then*

$$\boldsymbol{p}_A(\alpha) = \sum_{n=1}^{\infty} \boldsymbol{p}_A(\alpha_n).$$

4. One may think in view of (8) that the infinitistic axiom of σ-additivity is superfluous. However, theorem (8) does not seem to be an adequate generalization of axiom III of additivity. One may try to generalize axiom III in a logical fashion not using the notion of $\boldsymbol{p}$-convergency. So, we turn out to certain Tarski's constructions [3].

The sequence $\{\alpha_n\}_{n=1,2,\ldots}$ is called *decreasing* or *increasing over the set* A if and only if, for $n = 1, 2, \ldots$, $\alpha_{n+1} \in C_A(\alpha_n)$ or $\alpha_n \in \boldsymbol{C}_A(\alpha_{n+1})$, respectively.

The sequence $\{\alpha_n\}_{n=1,2,\ldots}$ is called $\wedge$*-convergent over the set* A to the sentence α if and only if two following conditions are satisfied:

(a) $\alpha_n \in \boldsymbol{C}_A(\alpha)$ for $n = 1, 2, \ldots$;

(b) if $\alpha_n \in \boldsymbol{C}_A(\gamma)$ for $n = 1, 2, \ldots$, then $\alpha \in \boldsymbol{C}_A(\gamma)$.

The sequence $\{\alpha_n\}_{n=1,2,\ldots}$ is called $\vee$*-convergent over the set* A to the sentence α if and only if two following conditions are satisfied:

(a) $\alpha \in \boldsymbol{C}_A(\alpha_n)$ for $n = 1, 2, \ldots$;

(b) if $\gamma \in \boldsymbol{C}_A(\alpha_n)$ for $n = 1, 2, \ldots$, then $\gamma \in \boldsymbol{C}_A(\alpha)$.

Using the notions just introduced it is possible to formulate thcre equivalent versions of the infinitistic axiom in the theory of logical probability (III$^\infty$, III*, III**).

III$^\infty$. Suppose that the set A is consistent and the sequence $\{a_n\}_{n=1,2\ldots}$ is $\vee$-convergent over A to the sentence a. If the sets $A^{(a_i \wedge a_j)}$ are inconsistent for $i \neq j$, then $\boldsymbol{p}_A(a) = \sum_{n=1}^{\infty} \boldsymbol{p}_A(a_n)$.

III*. If the set A is consistent and the sequence $\{a_n\}_{n=1,2\ldots}$ is increasing and $\wedge$-convergent to the sentence a, then $\boldsymbol{p}_A(a) = \lim_{n=\infty} \boldsymbol{p}_A(a_n)$.

III**. If the set A is consistent and the sequence $\{a_n\}_{n=1,2\ldots}$ is decreasing and $\vee$-convergent to the sentence a, then $\boldsymbol{p}_A(a) = \lim_{n=\infty} \boldsymbol{p}_A(a_n)$.

Axiom III$^\infty$ of σ-additivity is a logical generalization of axiom III. Axioms III* and III** may be called the *axioms of* $\wedge$*-continuity* and of $\vee$*-continuity*, respectively.

It is a matter of routine to prove on the assumption of I, II, III that axioms III$^\infty$, III*, III** are mutually equivalent.

5. To clarify the relation between axiom III$^\infty$ and theorem (8) two following theorems may be presented.

(9) *Assume* I, II *and* III. *If the set A is consistent and the sequence* $\{a_1 \vee \ldots \vee a_n\}_{n=1,2\ldots}$ *is* $\boldsymbol{p}$*-convergent (for some* $\boldsymbol{p}$*) over A to the sentence a, then the sequence* $\{a_n\}_{n=1,2\ldots}$ *is* $\vee$*-convergent over A to the sentence a.*

(10) *Assume* I, II *and* III$^\infty$. *If the set A is consistent and the sequence* $\{a_n\}_{n=1,2\ldots}$ *is* $\vee$*-convergent over A to the sentence a, then the sequence* $\{a_1 \vee \ldots \vee a_n\}_{n=1,2\ldots}$ *is* $\boldsymbol{p}$*-convergent (for every* $\boldsymbol{p}$*) over A to the sentence a.*

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