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SPACES $C(X)$ AND $L(\mu)$ AS DIRECT OR INVERSE LIMITS IN THE CATEGORY OF BANACH SPACES

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1. Preliminaries. T will denote a directed (upward filtering) set with some (partial) order $\leqslant$. An *inverse system* in a category is a family of morphisms

$$\pi_t^s\colon B_s \to B_t, \qquad t \in T,\ s \in T,\ t \leqslant s, \tag{1}$$

such that $\pi_t^r = \pi_t^s \pi_s^r$ if $t \leqslant s \leqslant r$ and $\pi_t^t = \mathrm{id}_{B_t}$. An *inverse limit* of such a system is an object B_∞ together with morphisms $\pi_t\colon B_\infty \to B_t$ ($t \in T$) such that

(i) $\pi_t^s \pi_s = \pi_t$ for each $t \leqslant s$,

(ii) for each object F and each family of morphisms $\alpha_t\colon F \to B_t$ satisfying $\pi_t^s \alpha_s = \alpha_t$ for $t \leqslant s$ there is a unique morphism $\alpha\colon F \to B_\infty$ such that $\pi_t \alpha = \alpha_t$ for t in T.

In symbol: $B_\infty = \lim^{\leftarrow}(B_t, \pi_t^s)$; if $(\pi_t^s)_{t \leqslant s}$ is understood, we may write simply $\lim^{\leftarrow} B_t$.

Direct systems and *direct limits* are defined dually; we write $B^\infty = \lim^{\rightarrow} B_t$.

In this paper, whenever we speak of direct or inverse limits of Banach spaces, we shall mean the limits in the category $\mathbf{Ban}_1$ of Banach spaces (over a field $\boldsymbol{F}$, where $\boldsymbol{F}$ is either $\boldsymbol{R}$ or $\boldsymbol{C}$), the morphisms being linear contractions (i.e. linear operators φ with $\|\varphi\| \leqslant 1$). It is well known that each inverse system in $\mathbf{Ban}_1$ has a Banach space B_∞ as an inverse limit, which may be constructed as a subspace of the l^∞-product $\prod_T^\infty B_t$. Specifically, $\lim^{\leftarrow} B_t$ consists of all bounded threads; a *thread* is a family $b = (b_t)_{t \in T}$ such that $b_t = \pi_t^s(b_s)$ for $t \leqslant s$; a thread is *bounded* iff $\|b\| = \sup \|b_t\| < \infty$. Also each direct system in $\mathbf{Ban}_1$ has a Banach space B^∞ as a direct limit, which can be constructed as a quotient of the l^1-sum $\sum_T^1 B_t$.

If B_1, B_2 are Banach spaces, then $B_1 \equiv B_2$ will denote that B_1 is $\mathbf{Ban}_1$-

isomorphic (i.e., linearly isometric) to B_2. If $1 \leqslant p \leqslant \infty$, L^p will mean $L^p(I)$, $I = [0, 1]$ with $\|f\|_p = (\int_0^1 |f(x)|^p dx)^{1/p}$ for $p < \infty$, $\|f\|_\infty = \operatorname{ess\,sup}|f|$; l_n^p $(n = 1, 2, \ldots)$ is $L^p(\mu)$ where μ is the counting measure on $\{1, \ldots, n\}$.

If $1 < p \leqslant \infty$ and $q \nearrow p$, then $L^p \equiv \lim^{\leftarrow} L^q$, the morphisms $L^r \to L^q$ $(r \geqslant q)$ being the identical injections (note that $\|f\|_r \geqslant \|f\|_q$). If $1 \leqslant p < \infty$ and $\varepsilon \searrow 0$, then $L^p \equiv \lim^{\rightarrow} L^{p+\varepsilon}$. These examples are special cases of the following general statement (cf. [4], # 11.8.3):

PROPOSITION 1. *If $(F_t)_{t\in T}$ is a family of Banach spaces, $F_s \subset F_t$ for $s \geqslant t$ (inclusion of vector spaces) and $\|f\|_s \geqslant \|f\|_t$ for $f \in F_s$, then the space*

$$F_\infty = \{f \in \bigcap_{t\in T} F_t\colon\ \|f\| = \sup_{t\in T}\|f\|_t < \infty\}$$

together with the identical injections $F_\infty \to F_t$ is an inverse limit of $(F_t)_{t\in T}$.

If $(G_t)_{t\in T}$ is a family of Banach spaces, $G_s \supset G_t$ for $s \geqslant t$ and $\|g\|_s \leqslant \|g\|_t$ for $g \in G_t$, let

$$G = \bigcup_{t\in T} G_t, \qquad \|g\| = \lim \|g\|_t, \qquad M = \{g \in G\colon\ \|g\| = 0\};$$

then the completion of G/M is a direct limit of $(G_t)_{t\in T}$.

PROPOSITION 2 (see [4], # 12.5.4(A)). *The contravariant conjugate-space functor from* $\mathbf{Ban}_1$ *to* $\mathbf{Ban}_1$ *defined as*

$$B \mapsto B^*, \qquad (\varphi\colon A \to B) \mapsto (\varphi^*\colon B^* \to A^*) \tag{2}$$

sends direct limits to inverse limits.

Yet, (2) does not send inverse limits to direct limits, in general. For example, if $p \to \infty$, then $\lim^{\leftarrow} L^p \equiv L^\infty$ but $\lim^{\rightarrow}(L^p)^* \equiv L^1$ and L^1 is not isomorphic to $(L^\infty)^*$.

If each B_t is separable, then $\lim^{\rightarrow} B_t$ is separable too; on the other hand, $\lim^{\leftarrow} B_t$ need not be separable.

For each direct system in $\mathbf{Ban}_1$, its direct limit in $\mathbf{Ban}_1$ is a completion of its direct limit in the category of normed vector spaces and linear contractions. J. S. Pym [3] studied conditions under which both direct limits coincide; e.g. if X is locally compact, then the space $\mathcal{M}(X)$ of bounded Radon measures on X is a direct limit — in either sense — of the system of subspaces

$$L_\mu = \{\nu \in \mathcal{M}(X)\colon\ \nu \ll \mu\}, \qquad \mu \in \mathcal{M}(X),\ \mu \geqslant 0.$$

2. Examples. E_{2^n} will denote the 2^n-dimensional vector space spanned by 2^n first Haar (or Walsh) functions (functions equal a.e. being identified); equivalently, E_{2^n} is the space spanned by the characteristic functions of the intervals

$$(3)\qquad I_k^n = \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right), \quad k = 1, 2, \ldots, 2^n.$$

$(E_{2^n})_{n=1,2,\ldots}$ is a direct system with inclusions $E_{2^n} \to E_{2^m}$, $n \leqslant m$. $E_{2^n}^1$ will denote E_{2^n} with the L^1-norm $\| \ \|_1$. Clearly, $E_{2^n}^1 \equiv l_{2^n}^1$ and, by Proposition 1, $\lim^{\to} E_{2^n}^1 \equiv L^1$. This gives a representation of L^1 as $\lim^{\to} l_{2^n}^1$.

$E_{2^n}^\infty$ will denote E_{2^n} with the L^∞-norm $\| \ \|_\infty$. Thus

$$(4)\qquad E_{2^n}^\infty \equiv l_{2^n}^\infty \quad \text{and} \quad \lim{}^{\to} E_{2^n}^\infty \equiv C(2^\omega),$$

where 2^ω is the Cantor set (this follows from Proposition 1 combined with the Stone–Weierstrass theorem).

The subspaces $E_{2^n}^1$ of L^1 defined above form also a natural inverse system

$$(5)\qquad E_{2^n}^1 \xleftarrow{\pi_n^m} E_{2^m}^1, \qquad n \leqslant m,$$

defined as follows: Let π_n: $L^1 \to E_{2^n}^1$ be the projection obtained by averaging on each interval (3), $k = 1, 2, \ldots, 2^n$, i.e., $\pi_n(f) = f_n$ where

$$f_n(x) = \frac{1}{|I_k^{(n)}|} \int_{I_k^{(n)}} f(u)\,du \quad \text{for } x \text{ in } I_k^{(n)}.$$

The map π_n^m is the restriction of π_n to $E_{2^m}^1$. A thread for the inverse system (5) is a sequence $(f_n)_{n=1,2,\ldots}$ such that $f_n \in E_{2^n}^1$ and $\pi_n^m(f_m) = f_n$ for $n \leqslant m$. This condition means that a thread (f_n) is a martingale relative to $\mathscr{F}_n$, where $\mathscr{F}_n$ is the smallest σ-algebra for which all such $f_1, \ldots, f_n$ are $\mathscr{F}_n$-measurable, i.e., the σ-algebra generated by the intervals (3).

Now, suppose that we want to test whether L^1 with projections π_n is an inverse limit of (5). We have to consider an arbitrary Banach space F and linear operators α_n: $F \to E_{2^n}^1$ such that $\|\alpha_n\| \leqslant 1$ and $\pi_n^m \alpha_m = \alpha_n$ for $n \leqslant m$. We ask whether there is an α: $F \to L^1$ such that $\alpha_n = \pi_n \alpha$. We may argue as follows: Let $f \in F$. Denote $g_n = \alpha_n(f)$. Then $g_n \in L^1$, $\sup\|g_n\|_1 < \infty$, $\pi_n^m(g_m) = g_n$ and $\|g_n\|_1 \leqslant \|g_m\|_1$ for $n \leqslant m$. Therefore, by a theorem on convergence of martingales (see [1]), the limit

$$g_\infty(x) = \lim g_n(x)$$

exists a.e. on I. Denote $\alpha(f) = g_\infty$. Then $\|\alpha(f)\|_1 = \|g_\infty\|_1 \leqslant \lim\|g_n\| \leqslant \|f\|$. Thus, we get a linear operator α: $F \to L^1$ with $\|\alpha\| \leqslant 1$.

Yet, $\pi_n g_\infty = g_n$ need not hold; thus, $\pi_n \alpha \neq \alpha_n$ in general. In fact (see [1]), the condition $\pi_n g_\infty = g_n$ for $n = 1, 2, \ldots$ holds if and only if $(g_n)_{n=1,2,\ldots}$ is uniformly integrable (equivalently, $\|g_\infty - g_n\|_1 \to 0$, or $\|g_\infty\|_1 = \lim\|g_n\|_1$).

Suppose, for instance, that $g_0 = 0$ and $g_n(x)$ is 2^{n-1} for $0 \leqslant x \leqslant 2^{-n}$, -2^{n-1} for $1-2^{-n} \leqslant x \leqslant 1$, and zero otherwise. Then $g_0, g_1, \ldots$ is a martingale, $\pi_n^m(g_m) = g_n$, $\|g_n\|_1 \leqslant 1$, but g_∞ vanishes a.e. If we define linear contractions α_n: $F \to E_{2^n}^1$ as $\alpha_n(t) = tg_n$ for t in F, then these operators cannot be factored through a single operator α: $F \to L^1$. Thus, L^1 is not an inverse limit of the subspaces (5).

Actually, $\lim^{\leftarrow}(E_{2^n}^1, \pi_n^m)$ may be identified with the space $\mathscr{M}(\mathbf{2}^\omega)$ of all Radon measures on $\mathbf{2}^\omega$. Indeed, $E_{2^n}^1 \equiv l_{2^n}^1$. Dualizing, we get a direct system

$$(\pi_n^m)^*\colon (E_{2^n}^1)^* \to (E_{2^m}^1)^*, \quad n \leqslant m, \quad \text{with } (E_{2^n}^1)^* \equiv l_{2^n}^\infty.$$

The spaces $(E_{2^n}^1)^*$ can be regarded as subspaces of L^∞, coinciding with $E_{2^n}^\infty$; it can be verified that $(\pi_n^m)^*$ are identical embeddings. Hence, by (4) and by Proposition 2, we get

$$\lim^{\leftarrow} E_{2^n}^1 \equiv \lim^{\leftarrow}(E_{2^n}^1)^{**} \equiv \left(\lim^{\rightarrow}(E_{2^n}^1)^*\right)^* \equiv C(\mathbf{2}^\omega)^* \equiv \mathscr{M}(\mathbf{2}^\omega).$$

Under this identification of $\lim^{\leftarrow} E_{2^n}^1$ with $\mathscr{M}(\mathbf{2}^\omega)$, the thread $g_0, g_1, g_2, \ldots$ considered above corresponds to the measure $\frac{1}{2}(\delta_0 - \delta_1)$, where δ_x denotes the Dirac measure at x, $0 = (0, 0, \ldots) \in \mathbf{2}^\omega$, $1 = (1, 1, \ldots) \in \mathbf{2}^\omega$.

Similarly, if we consider the inverse system π_n^m: $E_{2^m}^\infty \to E_{2^n}^\infty$, $n \leqslant m$, with the averaging operators π_n^m, we get

$$\lim^{\leftarrow} E_{2^n}^\infty \equiv (\lim^{\rightarrow} E_{2^n}^1)^* \equiv (L^1)^* \equiv L^\infty.$$

Note that $\lim^{\leftarrow} E_{2^n}^\infty \equiv L^\infty$ also follows from the martingale theorem, as the involved functions are uniformly bounded.

Now, let p_n^m: $E_{2^m}^\infty \to E_{2^n}^\infty$, $n \leqslant m$, be projections defined as follows: if $f \in E_{2^m}^\infty$, then $p_n^m(f)$ is defined on I_k^n as the value of f on that sub-interval I_j^m of I_k^n which is the farthest to the left. Then $E_{2^n}^\infty$ may be regarded as a product of 2^n copies of the field F (in $\mathbf{Ban}_1$); specifically, if $f \in E_{2^n}^\infty$ and a_k is the value of f on the interval I_k^n ($k = 1, \ldots, 2^n$), then f corresponds to the sequence $(a_1, \ldots, a_{2^n})$ in $\prod_{2^n}^\infty F$. The map p_n^m corresponds to the canonical projection from $\prod_{2^m}^\infty F$ onto $\prod_{2^n}^\infty F$. Consequently, $\lim^{\leftarrow}(E_{2^n}^\infty, p_n)$ is a product of ω copies of F, i.e., it is isometrically isomorphic to l^∞.

3. Conjugate L-spaces and $C(X)$ as inverse limits of $l_{n(t)}^1$ and $l_{n(t)}^\infty$.

LEMMA 1. *Each L-space is a direct limit of a directed family of finite-dimensional subspaces E_γ such that $E_\gamma \equiv l_{n(\gamma)}^1$.*

This is a well-known consequence of the Kakutani–Maharam representation of an L-space in the form

$$(6) \qquad l^1(S) + \sum_{\xi \in \Xi}^{1} L^1(2^{U_\xi}),$$

where Ξ and S are sets (possibly empty) and U_ξ ($\xi \in \Xi$) are infinite; $\sum^1$ and $+$ denote $\mathbf{Ban}_1$-coproducts (i.e. l^1-sums), 2^{U_ξ} denotes the generalized Cantor set with the product λ_ξ of copies of $(\frac{1}{2}, \frac{1}{2})$-measures ([4], #26.4.7).

LEMMA 2. *Let $\pi_t^s: A_s \to A_t$ ($t, s \in T$, $t \leqslant s$) be an inverse system in the category* $\mathbf{Ban}_1$ *such that $A_t = B_t^*$ for t in T and let each π_t^s be *weakly continuous. Then* $\lim^{\leftarrow} A_t \equiv B^*$ *for some Banach space B.*

Proof. There exist $\varphi_t^s: B_t \to B_s$ such that $(\varphi_t^s)^* = \pi_t^s$. We claim that $\varphi_s^r \varphi_t^s = \varphi_t^r$ if $t \leqslant s \leqslant r$. Indeed, from $\pi_t^r = \pi_t^s \pi_s^r$ we get $(\pi_t^r)^* = (\pi_s^r)^* (\pi_t^s)^*$, i.e., $(\varphi_t^r)^{**} = (\varphi_s^r)^{**} (\varphi_t^s)^{**}$. Composing these maps with the canonical maps $\varkappa_t: B_t \to B_t^{**}$ and making use of the naturality relation $\varkappa_s \varphi_t^s = (\varphi_t^s)^{**} \varkappa_t$ we get $\varphi_t^r = \varphi_s^r \varphi_t^s$. Thus, $(\varphi_t^s)_{t \in T}$ is a direct system, with a direct limit B. By Proposition 2, $\lim^{\leftarrow} A_t$ is isomorphic (in $\mathbf{Ban}_1$) to B^*.

COROLLARY 1. *Let π_t^s be an inverse system in* $\mathbf{Ban}_1$ *and let each A_t be reflexive. Then* $\lim^{\leftarrow} A_t$ *is a conjugate space (but it need not be reflexive, e.g.,* $L^\infty = \lim^{\leftarrow} L^p$).

COROLLARY 2. *If* $\dim A_t < \infty$ *for each t, then* $\lim^{\leftarrow} A_t$ *is a conjugate space.*

THEOREM 1. *An L-space is a conjugate space if and only if it can be represented as an inverse limit of finite-dimensional spaces $l^1_{n(t)}$, $t \in T$.*

THEOREM 2. *The space $C(X)$ of continuous functions on a compact space X is a conjugate space if and only if it can be represented as an inverse limit of finite-dimensional spaces $l^\infty_{n(t)}$, $t \in T$.*

Proof. In both cases, the "if" part follows from Corollary 2. Now, suppose that $L^1(\mu) \equiv B^*$, where B is a Banach space. By a theorem of Lazar and Lindenstrauss [2], there is a directed family (B_t) of subspaces of B, ordered by inclusion, such that $\dim B_t < \infty$, $B_t \equiv l^\infty_{n(t)}$ for $t \in T$ and $\bigcup B_t$ is dense in B. Thus, B is a direct limit of (B_t) and hence B^* is an inverse limit of $l^1_{n(t)}$.

Now, let $C(X)$ be a conjugate space A^*. By a result of Grothendieck (see e.g. [4], #27.4.1), $A \equiv L^1(\mu)$. Lemma 1 yields $A \equiv \lim^{\rightarrow} l^1_{n(t)}$. Hence, by Proposition 2, $A^* \equiv \lim^{\leftarrow} l^\infty_{n(t)}$.

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