COLLOQUIUM MATHEMATICUM

VOL. XVII 1967 FASC. 2

ON STOCHASTIC PROCESSES OVER AN ABELIAN LOCALLY COMPACT GROUP

BY

R. JAJTE (ŁÓDŹ)

Let Ω be a set with a σ-field $\mathfrak{B}$ of subsets and with a σ-measure p defined on $\mathfrak{B}$ and let $p(\Omega) = 1$. Let T be an abelian locally compact group and e the unit element of the group T. We denote by T^* the group of characters χ of the group T. The unit character is denoted by $\mathbf{1}$.

A stochastic process over a group T is given if to every $t \in T$ a $\mathfrak{B}$-measurable function (with complex values) defined on Ω is assigned:

$$\xi_t = \xi(t, \omega), \qquad t \in T, \omega \in \Omega.$$

Definition 1. A process $\xi = \xi(t, \omega)$ is called a *stationary process* (in the broad sense), see [1] and [2], if

1° $\langle \xi_t \rangle = m = \mathrm{const}$ $(t \in T)$ (1);

2° $\xi_t \in L_2(\Omega, p)$ $(t \in T)$;

3° $\langle \xi_t \bar{\xi}_s \rangle = b(ts^{-1})$, where $b(t)$ is a continuous function defined in T.

In what follows we will be dealing only with stationary processes. The function $b(t)$ appearing in 3° is called the *correlation function* of the process ξ. It is easily verified that the function $b(t)$ is positively definite (2). By the theorem of Weil [5] there exists a finite positive Radon measure μ over T^* such that

$$b(t) = \int_{T^*} \chi(t) \mu(d\chi). \tag{1}$$

The measure μ is uniquely determined by the function b. This measure will be called the *spectral measure* of the process ξ. Denote by H_ξ

(1) $\langle \eta \rangle \overset{\text{def}}{=} \int_\Omega \eta(\omega) p(d\omega)$.

(2) i.e. for arbitrary elements $t_1, \ldots, t_N$ of the group T and complex numbers $l_1, \ldots, l_N$ we have

$$\sum_{i,j=1}^{N} b(t_i t_j^{-1}) l_i \bar{l}_j \geqslant 0.$$

the Hilbert space which is defined to be a subspace of the space $L_2(\Omega, p)$ spanned over the elements of the process ξ_t $(t \in T)$.

The process ξ defines in the space H_ξ a unitary representation of the group T. Namely, the suitable unitary operators U_s are defined by the formula

$$U_s \xi_t = \xi_{st}, \qquad t, s \in T$$

(for the remaining points of the space H_ξ the operators U_s are defined by a natural extension). By the generalized theorem of Stone (see e.g. [3]) we have for the operators U_s the spectral representation

$$U_s = \int_{T^*} \chi(s) E(d\chi), \tag{2}$$

where $E(A)$ is a regular, normed (³) and orthogonal (⁴) spectral family of projectors defined on the field of Borel subsets of T^*. Putting

$$F(A) = E(A) \xi_e \tag{3}$$

we get a spectral representation

$$\xi_t = \int_{T^*} \chi(t) F(d\chi). \tag{4}$$

of the process (see [1] and [2]). The function $F(A)$, with values from the Hilbert space, is called a *random measure* of the process ξ. There exists a close relation between the spectral and the random measures:

$$\langle |F(A)|^2 \rangle = \mu(A). \tag{5}$$

In fact, we have

$$b(s) = \int_{T^*} \chi(s) \mu(d\chi) = \langle \xi_s \xi_e \rangle = \Big\langle \int_{T^*} \chi(s) F(d\chi) \overline{\int_{T^*} \chi(e) F(d\chi)} \Big\rangle,$$

but $\chi(e) \equiv 1$, thus making use of an easily proved formula (⁵)

$$\Big\langle \int_{T^*} f(\chi) F(d\chi) \overline{\int_{T^*} g(\chi) F(d\chi)} \Big\rangle = \int_{T^*} f(\chi) \overline{g(\chi)} \langle |F(d\chi)|^2 \rangle,$$

we have

$$b(s) = \int_{T^*} \chi(s) \langle |F(d\chi)|^2 \rangle. \tag{*}$$

By the uniqueness of the spectral measure μ, (*) yields formula (5).

(³) i.e. $E(T) = I$, where I is the unitary operator.

(⁴) i.e. $E(A).E(B) = E(A \cap B)$.

(⁵) For simple functions this formula easily follows from the orthogonality of the random measure $F(A)$:

$$F(A) F(B) = \Theta \quad \text{for} \quad A \cap B = \emptyset.$$

Definition 2. Let a function f with the values from a linear metric space X (with the metric $|\alpha-\beta|$) be defined on the group T. Then function f is said to possess the *mean value over the group T equal to m* if for every $\varepsilon>0$ there exists a system of elements of the group $t_1, \ldots, t_N$ such that

$$\sup_{t\in T}\left|\frac{1}{N}\sum_{i=1}^{N} f(tt_i)-m\right|<\varepsilon .$$

The point m of the space X is then uniquely defined. This point will be denoted by $M_T f$ (see e.g. [4]).

Now we will prove a theorem which is an analog of the classical law of great numbers for stationary processes [1].

THEOREM. *If ξ is a stochastic stationary process (definition 1), then*

$$M_T\xi = F(\{1\}). \tag{6}$$

Obviously, the process is regarded in this formula as a function over T with values from the Hilbert space $L_2(\Omega, p)$.

Proof. To prove formula (6) it suffices to show that

$$M_T \int_{T^*\setminus\{1\}} \chi(t)F(d\chi) = \Theta, \tag{7}$$

where Θ is the null random variable (vanishing almost everywhere on Ω). Let $\|\cdot\|$ be the norm in $L_2(\Omega, p)$. We have

$$\begin{aligned}
&\left\|\frac{1}{N}\sum_{i=1}^{N}\int_{T^*\setminus\{1\}} \chi(tt_i)F(d\chi)\right\|^2 \\
&= \frac{1}{N^2}\left\langle \sum_{k=1}^{N}\int_{T^*\setminus\{1\}}\chi(tt_k)F(d\chi)\sum_{l=1}^{N}\int_{T^*\setminus\{1\}}\chi(tt_l)F(d\chi)\right\rangle \\
&= \frac{1}{N^2}\int_{T^*\setminus\{1\}}\sum_{k,l=1}^{N}\chi(tt_k)\overline{\chi(tt_l)}\,\mu(d\chi) \\
&= \int_{T^*\setminus\{1\}}\left|\frac{1}{N}\sum_{k=1}^{N}\chi(t_k)\right|^2\mu(d\chi).
\end{aligned} \tag{8}$$

Thus it suffices to show that the last expression in formula (8) can be made as we please, the elements $t_1, \ldots, t_N$ of the group being chosen suitably. To this aim consider a compact set $Z\subset T^*\setminus\{1\}$. For every $\chi\in Z$ there exists a $t_\chi\in T$ such that $\chi(t_\chi)\neq 1$. Denote by $O(\chi, t_\chi)$ the neighbourhood of the character χ given by the formula

$$O(\chi, t_\chi) = \left\{\hat\chi\colon |\hat\chi(t_\chi)-\chi(t_\chi)| < \frac{|1-\chi(t_\chi)|}{2}\right\}.$$

Obviously

$$\bigcup_{\chi\in Z} O(\chi, t_\chi) \subset Z.$$

There exists in Z a finite system of characters $\chi_1, \ldots, \chi_m$ such that

$$Z \subset \bigcup_{i=1}^{m} O(\chi_i, t_{\chi_i}).$$

Let $\eta > 0$. Put $t_{\chi_i} = t_i$ and $Z_i = O(\chi_i, t_i) \cap Z$. Let $\hat{Z}_\nu$ be a compact set included in Z_ν such that $\mu(Z_\nu \setminus \hat{Z}) < \varepsilon/3m$. Let us fix for every $\hat{Z}_\nu$, an N_ν such that

$$\left|\frac{1}{N_\nu}\sum_{j=1}^{N_\nu}\chi(t_\nu^j)\right| \geqslant \eta$$

outside the set $\tilde{Z}_\nu$ for which $\mu(Z_\nu \setminus \tilde{Z}_\nu) < \varepsilon/3m$ and consider the product

$$\pi(\chi) = \prod_{\nu=1}^{m} \frac{1}{N_\nu}\sum_{j=1}^{N_\nu}\chi(t_\nu^j).$$

On the set $\tilde{Z} = \bigcup_{\nu=1}^{m} \tilde{Z}_\nu$ we have $|\pi(\chi)| < \eta$. Moreover

$$\pi(\chi) = \frac{1}{N_1 \ldots N_\nu} \sum_{(j_1,\ldots,j_m)} \chi(t_1^{j_1} \ldots t_m^{j_m}).$$

Thus, after renumerating the elements of the forms $t_1^{j_1} \ldots t_m^{j_m}$ we can write

$$\pi(\chi) = \frac{1}{N}\sum_{k=1}^{N}\chi(t_k),$$

where $N = N_1 N_2 \ldots N_m$.

Now we write the integral in formula (8) in another form:

$$\sigma = \int\limits_{T^*\setminus\{1\}} \left|\frac{1}{N}\sum_{k=1}^{N}\chi(t_k)\right|^2 = \int\limits_{T^*\setminus\{1\}\setminus Z} + \int\limits_{Z} = \sigma_1 + \sigma_2.$$

Let us fix a set Z such that $\mu(T^*\setminus\{1\}\setminus Z) < \varepsilon$ and then apply our argument to the expression under the integral sign. Hence

$$|\sigma| \leqslant |\sigma_1| + |\sigma_2| \leqslant \varepsilon + \eta\mu(\tilde{Z}) + \tfrac{2}{3}\varepsilon,$$

which completes the proof.

A process ξ is called *ergodic* if

$$M_T \xi = m = \langle \xi_t \rangle = \text{const}.$$

From formula (6) it follows that a process ξ is ergodic if and only if

(9) $$F(\{\mathbf{1}\}) = m \text{ almost everywhere on } \Omega.$$

Let us observe that formula (9) implies

(10) $$\langle |F(\{\mathbf{1}\})|^2\rangle = |m|^2$$

or, in other words, $\mu(\{\mathbf{1}\}) = |m|^2$.

Observe further that because of the stationarity of the process ξ we have $\langle F(\{\mathbf{1}\})\rangle = m$ and hence $\langle |F(\{\mathbf{1}\})-m|^2\rangle = |F(\{\mathbf{1}\})|^2-|m|^2$. Thus (10) implies (9). Therefore the process ξ is ergodic if and only if equation (10) holds for the spectral measure μ.

REFERENCES

[1] J. L. Doob, *Stochastic processes*, Now York — London 1953.

[2] J. Kampé de Fériet, *Analyse harmonique des fonctions aléatoires stationnaires d'ordre 2 sur un groupe abelien localement compact*, Comptes Rendus de l'Académie des Sciences de Paris 226 (1948), p. 868-870.

[3] R. Godement, *Sur une généralisation d'un théorème de Stone*, ibidem 218 (1944), p. 901-903.

[4] W. Maak, *Fastperiodische Funktionen*, Berlin-Göttingen-Heidelberg 1950.

[5] A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Paris 1940.

Reçu par la Rédaction le 9. 12. 1965