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UNIFORMIZATION OF A CLASS OF ALGEBRAIC FUNCTIONS OF THE THIRD DEGREE BY THE METHOD OF DIFFERENTIAL EQUATIONS

BY

J. JANIKOWSKI (ŁÓDŹ)

In this note we extend the method of uniformization used by Charzyński [1] to functions of a complex variable.

Consider the equation

$$f(z) = g(w), \tag{1}$$

where $f(z)$ and $g(w)$ are polynomials of the third degree. We will prove the following

THEOREM. *If for any pair (z_0, w_0) of complex numbers for which (1) holds true we have*

$$\text{either } f'(z_0) \neq 0 \text{ or } g'(w_0) \neq 0, \tag{2}$$

then there exist two meromorphic functions $z(t)$ and $w(t)$ defined over the whole open complex plane such that we have in this plane the equation

$$f\big(z(t)\big) = g\big(w(t)\big). \tag{3}$$

Proof. Consider the system

$$\frac{dz}{dt} = g'(w), \quad \frac{dw}{dt} = f'(z) \tag{4}$$

of differential equations with initial conditions

$$z(0) = z_0, \quad w(0) = w_0, \tag{5}$$

where (z_0, w_0) is any fixed pair of complex numbers satisfying (1) and t denotes a complex variable.

In a neighbourhood of $t = 0$ system (4) has exactly one solution $z(t)$, $w(t)$ that meets initial conditions (5) and is composed of holomorphic functions none of which is a constant ([2], pp. 369-372 and 374-375). Because of the initial conditions the functions $z(t)$ and $w(t)$ satisfy in this neighbourhood of $t = 0$ equation (3).

Let us consider an arbitrary regular curve C without right-hand end given by equation

$$t = t(\tau), \quad t(0) = 0, \quad 0 \leqslant \tau < \tau_0. \tag{6}$$

We shall consider analytic continuations ([3], pp. 231 and 237) of the pair of functions $z(t)$ and $w(t)$ along parts of the curve C corresponding to partial intervals $\langle 0, \tau_1)$ contained in the interval $\langle 0, \tau_0)$ of parameter τ. These continuations, if they exist, satisfy at points, where both functions are holomorphic, the system (4) ([2], p. 488) as well as equation (3).

Denote by $\langle 0, \tau^*)$ the largest of the described partial intervals for which there exist analytic continuations of functions $z(t)$ and $w(t)$. There are two possibilities: either $\tau^* = \tau_0$ or $\tau^* < \tau_0$. We shall prove that the second possibility cannot occur.

We start with some auxiliary considerations.

Note first that if a pair (z, w) of complex numbers satisfies equation (1), then either both numbers are finite or both are infinite. Next we shall prove

LEMMA 1. *For every pair (z^*, w^*) of finite complex numbers satisfying* (1) *and for every t^* there exists exactly one pair of functions $\hat{z}(t)$ and $\hat{w}(t)$ holomorphic in a neighbourhood of t^* and satisfying the initial conditions*

$$\hat{z}(t^*) = z^* \quad \textit{and} \quad \hat{w}(t^*) = w^* \tag{7}$$

such that for every pair (z, w) of complex numbers satisfying (1) *and sufficiently close to (z^*, w^*) there exists exactly one t in a neighbourhood of t^* for which we have*

$$z = \hat{z}(t) \quad \textit{and} \quad w = \hat{w}(t). \tag{8}$$

Proof. By the conditions of the Theorem at least one of the numbers $f'(z^*)$ and $g'(w^*)$ is different from zero. Suppose we have

$$g'(w^*) \neq 0. \tag{9}$$

In view of the theorem on the existence of an implicit function, if a pair (z, w) that is close to the pair (z^*, w^*) satisfies (1), then the variable w may be presented as a holomorphic function

$$w = \varphi(z) \tag{10}$$

of variable z such that we have $w^* = \varphi(z^*)$ and the equation

$$f(z) = g\big(\varphi(z)\big) \tag{11}$$

holds true in a neighbourhood of z^*.

Consider the differential equation

$$\frac{dz}{dt} = g'_w\big(\varphi(z)\big). \tag{12}$$

This equation has exactly one solution $z = \hat{z}(t)$ defined in a neighbourhood of t^* which satisfies the initial condition $\hat{z}(t^*) = z^*$. Because of (9) we have in a sufficiently small neighbourhood of t^* the identity

$$\frac{\hat{z}'(t)}{g'_w\big[\varphi\big(\hat{z}(t)\big)\big]} = 1. \tag{13}$$

Consequently, we have in the same neighbourhood the equivalent equation

$$\int_{z^*}^{z} \frac{dz}{g'_w\big(\varphi(z)\big)} = t - t^*. \tag{13'}$$

The integral on the left-hand side of (13′) is computed along any regular curve joining z and z^* and lying in a suitable neighbourhood of z^*. Equation (13′) determines t as a holomorphic function of z in some neighbourhood of z^*, which we may put in symbols as

$$t = \psi(z) \tag{14}$$

with $\psi'(z) \neq 0$ in this neighbourhood. The above mapping is one-to-one in a suitably chosen neighbourhood of z^*. Consequently, the inverse mapping $z = \psi^{-1}(t) = \hat{z}(t)$ is also one-to-one in a neighbourhood of t^*.

If we write $\hat{w}(t) = \varphi\big(\hat{z}(t)\big)$, then, taking into account relations (13′), (13) and (11), we easily check that the functions $\hat{z}(t)$ and $\hat{w}(t)$ satisfy equations (4) and initial conditions (7) and are uniquely determined in a neighbourhood of t^*.

There is a small neighbourhood of (z^*, w^*) such that for a given z there is only one w such that (z, w) is in this neighbourhood and satisfies (1). Now, because to given z there corresponds by (14) exactly one value of t in a neighbourhood of t^*, we finally conclude that for every pair (z, w) satisfying (1) from a suitable neighbourhood of (z^*, w^*) there exists only one value t from a neighbourhood of t^* such that $z = \hat{z}(t)$ and $w = \hat{w}(t)$. Lemma 1 is thus proved.

We shall now prove the analogous

LEMMA 1′. *For every triple of complex numbers t_j^* $(j = 1, 2, 3)$ not necessarily all different there exist three different pairs of functions*

$$z = z_j(t), \quad w = w_j(t) \quad (j = 1, 2, 3) \tag{15}$$

such that the functions $z_j(t)$ and $w_j(t)$ are meromorphic in a neighbourhood of t_j^*, have the only pole at t_j^*, and satisfy in a neighbourhood of t_j^* differen-

tial equations (4) *and equation* (3). *Moreover, for every pair* (z, w) *of finite numbers satisfying* (1) *and sufficiently close to the pair* (∞, ∞) *there is exactly one* j $(1 \leqslant j \leqslant 3)$ *and exactly one* t *in a neighbourhood of* t_j^* *such that we have*

$$z = z_j(t) \quad \textit{and} \quad w = w_j(t).$$

Proof. Note that in a neighbourhood of the point $z = \infty$ there exist exactly three different meromorphic functions

$$w = \varphi_j(z) = \beta_{-1j}z + \beta_{0j} + \frac{\beta_{1j}}{z} + \frac{\beta_{2j}}{z^2} + \ldots, \qquad \beta_{-1j} \neq 0 \qquad (j = 1, 2, 3) \tag{16}$$

satisfying the condition

$$f(z) = g\big(\varphi_j(z)\big) \qquad (j = 1, 2, 3). \tag{17}$$

Consider the differential equation

$$\frac{dz}{dt} = g'_w\big(\varphi_j(z)\big). \tag{18}$$

Substitute

$$u = \frac{1}{z}. \tag{19}$$

Because $g'(w)$ is a polynomial of the second degree, we may bring equation (18), after having implemented substitution (19), to the equivalent form

$$\frac{du}{dt} = \lambda_{0j} + \lambda_{1j}u + \lambda_{2j}u^2 + \ldots = \psi_j(u), \qquad \lambda_{0j} \neq 0. \tag{20}$$

For u sufficiently close to 0 we have $\psi_j(u) \neq 0$. So equation (20) may be written in the form

$$\frac{du}{dt} \cdot \frac{1}{\psi_j(u)} = 1. \tag{21}$$

Writing

$$\frac{1}{\psi_j(u)} = \mu_{0j} + \mu_{1j}u + \mu_{2j}u^2 + \ldots, \qquad \mu_{0j} \neq 0, \tag{22}$$

we get a relation

$$\mu_{0j}u + \frac{1}{2}\mu_{1j}u^2 + \frac{1}{3}\mu_{2j}u^3 + \ldots = t + \gamma_j \tag{23}$$

equivalent with (20), where γ_j is a constant. Denote by $u_j(t)$ a function satisfying equation (20) or the equivalent equation (23) with initial condition

$$u_j(t_j^*) = 0 \tag{24}$$

in a neighbourhood of t_j^* (we then obviously have $\gamma_j = -t_j^*$). We see that for every $j = 1, 2, 3$ there exists such a function in a neighbourhood of t_j^*, and, in view of the condition $\lambda_{0j} \neq 0$ of (20) we have

$$u_j'(t_j^*) \neq 0 \quad (j = 1, 2, 3). \tag{25}$$

Putting

$$z_j(t) = \frac{1}{u_j(t)}, \qquad w_j(t) = \varphi_j\big(z_j(t)\big) \tag{26}$$

we check with the aid of (17) and (18) that for every $j = 1, 2, 3$ the two functions (26) satisfy in a neighbourhood of t_j^* the identity

$$f\big(z_j(t)\big) = g\big(w_j(t)\big) \tag{17'}$$

and the differential equation

$$\frac{dz_j}{dt} = g_w'\big(w_j(t)\big), \tag{18'}$$

and, consequently, the system (4) of differential equations. Moreover, according to (25) and (16) the functions (26) have a single pole at t_j^*.

We see that there is a neighbourhood of the pair (∞, ∞) such that for every z there are in a neighbourhood of ∞ exactly three different numbers $\varphi_j(z)$, $j = 1, 2, 3$, such that $f(z) = g\big(\varphi_j(z)\big)$. Consequently, for every pair (z, w) in a neighbourhood of the pair (∞, ∞) satisfying (1) there exists a j ($= 1, 2$ or 3) such that $w = \varphi_j(z)$. On the other hand, to a given z there corresponds through the first of the functions (26) exactly one t in a neighbourhood of t_j^* such that we have $z = z_j(t)$. Thus for every pair (z, w) in a neighbourhood of (∞, ∞) satisfying (1) at least one of equations (15) is satisfied. However, if, for such a pair (z, w), we had two of these relations, say the first and the second, then we had, say,

$$z = z_1(t_1) = z_2(t_2) \quad \text{and} \quad w = w_1(t_1) = w_2(t_2).$$

In view of (26) we then had

$$\varphi_1(z) = \varphi_2(z).$$

Yet this is impossible because functions (16) being different are mutually different everywhere in a neighbourhood of ∞. This completes the proof.

We next prove

LEMMA 2. *If for a given pair* (z^*, w^*) *and a given regular curve* C *given by* $t = t(\tau)$ *there exists a sequence* $\{\tau_\nu\}$ *such that* $\tau_\nu \to \tau^*$ *while* $\tau_\nu < \tau^*$ *for every* ν, *and for which we have*

$$z\big(t(\tau_\nu)\big) \to z^* \quad \text{and} \quad w\big(t(\tau_\nu)\big) \to w^*, \tag{27}$$

then the functions $z(t)$ *and* $w(t)$ *admit a holomorphic continuation through the point* $t^* = t(\tau^*)$.

Proof. Since $f\big(z(t_\nu)\big) = g\big(w(t_\nu)\big)$ for $\nu = 1, 2, \ldots,$ where we have put t_ν for $t(\tau_\nu)$, the pair (z^*, w^*) satisfies (1). Because of (2) there exists in a neighbourhood of t^* of radius ϱ^*, say, exactly one solution $\tilde{z}(t)$ and $\tilde{w}(t)$ of the system (4) of differential equations satisfying initial conditions $\tilde{z}(t^*) = z^*$ and $\tilde{w}(t^*) = w^*$, and having properties described in Lemma 1. Consider a pair $\big(z(t_\nu), w(t_\nu)\big)$ that is sufficiently close to (z^*, w^*). In view of Lemma 1 there is in this neighbourhood of t^* exactly one point ϑ_ν such that

$$\tilde{z}(\vartheta_\nu) = z(t_\nu) \quad \text{and} \quad \tilde{w}(\vartheta_\nu) = w(t_\nu). \tag{28}$$

Consider the pair of functions

$$\tilde{z}(t+\vartheta_\nu - t_\nu) \quad \text{and} \quad \tilde{w}(t+\vartheta_\nu - t_\nu). \tag{29}$$

It follows from the properties of the functions $\tilde{z}(t)$ and $\tilde{w}(t)$ that ϑ_ν is close to t^* if t_ν is. Functions (29) are so defined and holomorphic in a neighbourhood of $t^* - (t_\nu - \vartheta_\nu)$ of radius not less than $\varrho^*/2$. Besides we may assume that this neighbourhood contains the points t_ν and t^* as well as the part C_ν^* of the curve C lying between these points. For $t = t_\nu$ the functions (29) take on the same values as the functions $z(t)$ and $w(t)$. In a neighbourhood of t_ν the functions (29) and the functions $z(t)$ and $w(t)$ fulfil the same system (4) of differential equations. Moreover, we have equations (28), which can be viewed as a kind of initial conditions. Because of the unicity of solutions of differential equations (4) we have in the considered neighbourhood of t_ν identically in t the equations

$$\tilde{z}(t+\vartheta_\nu - t_\nu) = z(t) \quad \text{and} \quad \tilde{w}(t+\vartheta_\nu - t_\nu) = w(t). \tag{30}$$

Thus we see that functions (29) constitute an analytic continuation of the functions $z(t)$ and $w(t)$ over a domain containing the closed curve C_ν^*. Therefore the functions $z(t)$ and $w(t)$ can be analytically continued from the point t_ν along a part of the curve C that contains t^* in its interior. This proves Lemma 2.

We now prove an analogous

LEMMA 2'. *If there is a sequence $\{\tau_\nu\}$ such that we have $\tau_\nu \to \tau^*$ while $\tau_\nu < \tau^*$ for every ν, for which*

$$z\big(t(\tau_\nu)\big) \to \infty \quad \text{and} \quad w\big(t(\tau_\nu)\big) \to \infty,$$

then the functions $z(t)$ and $w(t)$ can be continued meromorphically through the point $t^* = t(\tau^*)$.

Proof. In view of Lemma 1' there exist in a circular neighbourhood of t^* of radius ϱ, say, three pairs of functions

$$z_j(t) \quad \text{and} \quad w_j(t) \quad (j = 1, 2, 3) \tag{31}$$

that are meromorphic in this neighbourhood, have the only pole at t^* and fulfil differential equations (4) in a suitable neighbourhood of t^* and identity (3) in the whole neighbourhood of t^*. Consider a pair $\big(z(t_\nu), w(t_\nu)\big)$, where t_ν stands for $t(\tau_\nu)$, which is sufficiently close to (∞, ∞). It results from Lemma 1' that for exactly one j the functions (31) are so that in a neighbourhood of t^* there is precisely one point ϑ_ν for which we have

$$z_j(\vartheta_\nu) = z(t_\nu) \quad \text{and} \quad w_j(\vartheta_\nu) = w(t_\nu). \tag{32}$$

Consider the functions

$$z_j(t + \vartheta_\nu - t_\nu) \quad \text{and} \quad w_j(t + \vartheta_\nu - t_\nu). \tag{33}$$

We conclude from the properties of the functions $z(t)$ and $w(t)$ that ϑ_ν is close to t^* if t_ν is. Thus the functions (33) are defined and meromorphic in a neighbourhood of $t^* - (t_\nu - \vartheta_\nu)$ of radius not smaller than $\varrho/2$. Besides we may assume that this neighbourhood contains the points t_ν and t^* together with the part C_ν^* of the curve C, given by $t = t(\tau)$, lying between these points. For $t = t_\nu$ functions (33) take on the same values as $z(t)$ and $w(t)$. In a neighbourhood of t_ν functions (33) are holomorphic, fulfil the same system (4) of differential equations as do the functions $z(t)$ and $w(t)$, and, moreover, they satisfy equations (32) which we can treat as initial conditions. In view of the unicity of solutions of differential equations (4), we have in a neighbourhood of t_ν identically in t the equations

$$z_j(t + \vartheta_\nu - t_\nu) = z(t) \quad \text{and} \quad w_j(t + \vartheta_\nu - t_\nu) = w(t). \tag{34}$$

Thus we see that the meromorphic functions (33) defined in a neighbourhood of t^* are analytic continuations of the functions $z(t)$ and $w(t)$ over a domain containing the closed curve C_ν^* in its interior. This means that the functions $z(t)$ and $w(t)$ admit an analytic continuation from point t_ν along the part of the curve C that contains t^* as its inner point. This proves Lemma 2'.

Let us now return to the proof of the Theorem. Suppose we have $\tau^* < \tau_0$ for a regular curve C given by equation (6). Take into consideration any sequence $\{\tau_\nu\}$ such that $\tau_\nu \to \tau^*$ while $\tau_\nu < \tau^*$ for all ν. It is always possible to select from the two sequences $\{z(t(\tau_\nu))\}$ and $\{w(t(\tau_\nu))\}$ two subsequences, to be denoted for brevity's sake as the sequences themselves, which, according to the remark on p. 358, either converge both to finite limits or both tend to ∞. Lemmas 2 and 2' imply that in each case the functions $z(t)$ and $w(t)$ can be analytically continued along a part of the curve C containing $t^* = t(\tau^*)$ in its interior. This contradicts the definition of τ^*. Therefore we must have $\tau^* = \tau_0$. This, however, means that the functions $z(t)$ and $w(t)$ can be analytically continued in the whole open plane. They are, therefore, univalent meromorphic functions and fulfil system (4) of differential equations at every point of holomorphicity and thus equations (3) as well. Equations (3) hold true also at the poles. This concludes the proof.

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