COLLOQUIUM MATHEMATICUM

VOL. LIV 1987 FASC. 2

SOME BANACH TECHNIQUES IN VECTOR-VALUED FOURIER ANALYSIS

BY

JOSÉ L. RUBIO DE FRANCIA AND JOSÉ L. TORREA (MADRID)

1. Introduction and basic facts. Given a linear operator in some L^p-space, the question of extending it in a natural way to L^p_E, where E is a Banach space, appears quite often in probability theory and in the study of geometrical properties of Banach spaces. In this paper, we define and study a relation "$E < F$" between Banach spaces E and F which express their better or worse behaviour with respect to this type of extensions.

We shall write $L^p = L^p(\mathbf{R}, dx)$, $1 < p < \infty$, and denote by $\mathscr{L}(L^p)$ the space of all bounded linear operators in L^p. Given a Banach space E, we put (as in [23])

$$\mathscr{L}_E(L^p) = \{S \in \mathscr{L}(L^p):\ S_E = S \otimes \mathrm{Id}_E \text{ is bounded in } L^p_E\}.$$

We shall need the notions of (Rademacher) type and cotype of a Banach space (see [14]) and adopt the usual notation:

$$p(E) = \sup\{p:\ E \text{ is of type } p\},$$

$$q(E) = \inf\{q:\ E \text{ is of cotype } q\}.$$

DEFINITION. Given Banach spaces E and F we say that F *extends better than* E, and write $E < F$, if $\mathscr{L}_E(L^p) \subset \mathscr{L}_F(L^p)$ for all p $(1 < p < \infty)$.

In some cases, the (seemingly weaker) condition $\mathscr{L}_E(L^2) \subset \mathscr{L}_F(L^2)$ will also be considered instead of $E < F$. By the closed graph theorem, $E < F$ is equivalent to the existence of constants C_p, $1 < p < \infty$, such that

$$\sup\{\|S_F f\|_{L^p_F}:\ f \in L^p \otimes F,\ \|f\|_{L^p_F} \leqslant 1\} \leqslant C_p \sup\{\|S_E g\|_{L^p_E}:\ g \in L^p \otimes E,\ \|g\|_{L^p_E} \leqslant 1\}$$

for all $S \in \mathscr{L}(L^p)$. The following remarks are either obvious or easy consequences of known facts:

Remarks. 0. For $0 < r < \infty$, $l^r < L^r(\mu)$ for any measure space (Ω, A, μ).

1. If F is isomorphic to a subspace of a quotient of F, then $E < F$.

2. If F is finitely representable in E, then $E < F$.

3. $E < F \Leftrightarrow E^* < F^*$.

In fact, $\Rightarrow$ consists in a simple duality argument, and $\Leftarrow$ follows from Remark 2 because F^{**} is finitely representable in F (principle of local reflexivity).

4. $l^1 < F$ for every Banach space F.

This is easy to verify (see [22] or [23]).

5. Given $p > 1$, we have $l^p < l^q \Leftrightarrow p \leqslant q \leqslant 2$ or $2 \leqslant q \leqslant p$ (see [7]).

Finally, we remark that the choice of the measure space $(\boldsymbol{R}, dx)$ is rather immaterial, and nothing is changed if we take in our definition an interval or $\boldsymbol{R}^n$ with a measure $w(x)dx$ equivalent to Lebesgue measure.

2. Type and cotype results.

THEOREM 1. *Let E be a Banach space of type $p > 1$ and cotype q. If*

$$\mathscr{L}_E(L^2) \subset \mathscr{L}_F(L^2),$$

then F is also of type p and cotype q.

Before proving this theorem, let us point out several consequences.

COROLLARY 1. *If $p(E) > 1$ (i.e., if E is B-convex), then $E < F$ implies*

$$p(E) \leqslant p(F) \leqslant q(F) \leqslant q(E).$$

The restriction $p(E) > 1$ cannot be removed, but the case $p(E) = 1$ is actually the easiest to handle, since we have

COROLLARY 2. *For a Banach space E, the following statements are equivalent*:

(a) $p(E) = 1$ (*i.e., E is not B-convex*).

(b) $\mathscr{L}_E(L^2) \subset \mathscr{L}_F(L^2)$ *for every Banach space F.*

(c) $E < F$ *for every Banach space F.*

Proof. If $p(E) > 1$, Theorem 1 tells us that statement (b) is false for the Banach space $F = l^r$, $1 \leqslant r \leqslant p(E)$. On the other hand, (a) implies that l^1 is finitely representable in E (see [16]), and then by Remarks 2 and 4 we have $E < l^1 < F$ for every F.

COROLLARY 3. *For a Banach space F, the following statements are equivalent*:

(a) *F is isomorphic to a Hilbert space.*

(b) $\mathscr{L}_E(L^2) \subset \mathscr{L}_F(L^2)$ *for every Banach space E.*

(c) $E < F$ *for every Banach space E.*

Proof. If (a) holds, then $\mathscr{L}_F(L^p) = \mathscr{L}(L^p)$, $1 < p < \infty$ (see, e.g., [7]), and, consequently, (c) holds. On the other hand, (b) applied to $E = l^2$ implies (by Theorem 1) that F is of type 2 and cotype 2, which by a well-known theorem of Kwapień [9] is equivalent to (a).

Proof of Theorem 1. Let $(r_j(t))_{j=1}^{\infty}$ be the sequence of Rademacher functions in $[0, 1]$ and extend them to all of $\boldsymbol{R}$ by letting $r_j(t) = 0$ when $t \notin [0, 1]$. Assume that E is of type p with constant $T_p(E)$ and define s by $1/p = 1/s + 1/2$. For every $\alpha = (\alpha_j)_{j=1}^{\infty} \in l_+^s$ (i.e., $\alpha_j > 0$ for all j) we denote by S^α the following operator:

$$S^\alpha f(t) = \sum_j \alpha_j \Big(\int_{j-1}^{j} f \Big) r_j(t) \qquad (f \in L^2). \tag{1}$$

For functions $f \in L^2 \otimes E$, $S_E^\alpha f$ is formally defined by the same formula (1), and we have

$$\|S_E^\alpha f\|_{L_E^2} \leqslant T_p(E) \Big(\sum_j \|\alpha_j \int_{j-1}^{j} f\|_E^p\Big)^{1/p} \leqslant T_p(E) \|\alpha\|_s \|f\|_{L_E^2}.$$

Since we are assuming that $\mathscr{L}_E(L^2) \subset \mathscr{L}_F(L^2)$, there exists a constant $C > 0$ such that

$$\|S_F^\alpha f\|_{L_F^2} \leqslant C T_p(E) \|\alpha\|_s \|f\|_{L_F^2} \qquad (f \in L^2 \otimes F). \tag{2}$$

Now, given $b_1, b_2, \ldots, b_n \in F$, we apply (2) to the function

$$f_\alpha(t) = \sum_{j=1}^{n} \alpha_j^{-1} X_{[j-1,j]}(t) b_j$$

and we obtain

$$\Big\|\sum_{j=1}^{n} b_j r_j\Big\|_{L_F^2} \leqslant C T_p(E) \|\alpha\|_s \Big(\sum_{j=1}^{n} \alpha_j^{-2} \|b_j\|_F^2\Big)^{1/2}. \tag{3}$$

But the infimum for all $\alpha \in l_+^s$ of the expression on the right of (3) equals

$$C T_p(E) \Big(\sum_{j=1}^{n} \|b_j\|_F^p\Big)^{1/p},$$

and this proves that F is of type p with constant $T_p(F) \leqslant C T_p(E)$.

Assume now that E is of cotype q. Since $p(E) > 1$, Pisier's characterization of K-convexity (see [17]) implies that E^* is of type q' (where $1/q + 1/q' = 1$). By Remark 3 and the part already proved, it follows that F^* is of type q', and then F is of cotype q.

As we shall see later, the converse of Corollary 1 is not true in general, but the following partial converse does hold:

THEOREM 2. *Let E be a Banach space with $p(E) > 1$ and let $1 < r < \infty$. The following statements are equivalent*:

(a) $E < l^r$.

(b) $\mathscr{L}_E(L^2) \subset \mathscr{L}_{l^r}(L^2)$.

(c) $p(E) \leqslant r \leqslant q(E)$.

Proof. It is clear that (a) implies (b), and (b) implies (c) by Theorem 1, since $p(l^r) = \inf(r, 2)$ and $q(l^r) = \sup(2, r)$. Assume now that $p(E) \leqslant r \leqslant 2$. By the Maurey–Pisier theorem (see [14]), $l^{p(E)}$ is finitely representable in E, and we use Remarks 2 and 5 to obtain $E < l^{p(E)} < l^r$. Similarly, if $2 \leqslant r \leqslant q(E)$, then $l^{q(E)}$ is finitely representable in E and $E < l^{q(E)} < l^r$.

We denote by SQ_r the class of all Banach spaces which are isomorphic to a subspace of a quotient of some space $L^r(\mu)$.

THEOREM 3. *Given a Banach space F and $1 < r < \infty$, we have $l^r < F$ if and only if $F \in SQ_r$.*

Proof. If $F \in SQ_r$, then $L^r(\mu) < F$ for some measure μ (by Remark 1), and since $L^r(\mu)$ is finitely representable in l^r, it follows that $l^r < F$. Conversely, since every operator $S \in \mathscr{L}(L^r)$ has a bounded l^r-valued extension, $l^r < F$ implies $\mathscr{L}_F(L^r) = \mathscr{L}(L^r)$, and this is a characterization (due to Kwapień [10]) of the spaces $F \in SQ_r$.

A consequence of Theorem 3 is the following: If $l^r < F$ for some r $(1 < r < \infty)$, then F must be superreflexive. One can now give several counterexamples showing that the converse of Corollary 1 may fail.

EXAMPLE 1. Let F be the space defined in [8], which is of type 2 (therefore $q(F) < +\infty$) but not superreflexive. If we take $E = l^r$, $q(F) \leqslant r < \infty$, we have

$$p(E) = p(F) \leqslant q(F) \leqslant q(E),$$

but $E \not< F$.

EXAMPLE 2. The Schatten ideal $\mathscr{C}_r$ $(1 < r < 2)$ is of type r and cotype 2 (exactly the same as l^r), but it is known that $\mathscr{C}_r \notin SQ_r$, and therefore $l^r \not< \mathscr{C}_r$.

3. Further remarks and comments. If we define in the class of all Banach spaces the equivalence relation

$$E \sim F \text{ iff } E < F \text{ and } F < E,$$

one can order the equivalence classes in the obvious way and there exist a first element (consisting of the worst Banach spaces)

$$\mathscr{A}_1 = [l^1] = \{\text{Banach spaces which are not } B\text{-convex}\}$$

and a last element

$$\mathscr{A}_2 = [l^2] = \{\text{spaces isomorphic to some Hilbert space}\}.$$

Some intermediate classes are

$$\mathscr{A}_r = [l^r] = \begin{cases} \{E\colon E \in SQ_r \text{ and } p(E) = r\} & \text{if } 1 < r < 2, \\ \{E\colon E \in SQ_r \text{ and } q(E) = r\} & \text{if } 2 < r < \infty. \end{cases}$$

If $E \in \mathscr{A}_1$ and $F \in \mathscr{A}_2$, then for all p $(1 < p < \infty)$ we have

$$\mathscr{L}_E(L^p) = \{S \in \mathscr{L}(L^p) \text{ which are regular}\}, \quad \mathscr{L}_F(L^p) = \mathscr{L}(L^p),$$

where S regular means $|Sf| \leqslant A(|f|)$ for some positive operator $A \in \mathscr{L}(L^p)$. The first assertion is proved for instance in [4] and [23]. Each one of the extreme classes is characterized by a single operator in the following way. Let R denote the orthogonal projection of L^2 onto the span of $(r_j(t))_{j=1}^{\infty}$ and let $Tf = \hat{f}$ be the Fourier transform in L^2. Then

(i) $E \in \mathscr{A}_1$ if and only if $R \otimes \mathrm{Id}_E$ is not bounded in L_E^2 (see [17]);

(ii) $F \in \mathscr{A}_2$ if and only if $T \otimes \mathrm{Id}_F$ is bounded in L_F^2 (see [9]).

Finally, we make the following conjecture suggested by Theorems 1 and 2 and Corollaries 2 and 3:

CONJECTURE (**P 1337**). For arbitrary Banach spaces E and F one has $E < F$ if and only if $\mathscr{L}_E(L^2) \subset \mathscr{L}_F(L^2)$.

If the conjecture is true, one has in particular

$$E < l_E^2 \quad \text{for every Banach space } E. \tag{4}$$

Conversely, if case (4) were proved, one could prove the conjecture by using Maurey's theory of the factorization of operators. However, (4) may not be easy to obtain, since it would automatically imply a positive answer to Problem 2 in [10].

4. Applications.

I. Let G be an infinite compact abelian group and let Γ be its dual (discrete) group. Let us denote by $\mathscr{F}$ the mapping from $L^2(G)$ into $l^2(\Gamma)$ given by $f \to \mathscr{F}f = \hat{f}$, where

$$\hat{f}(\gamma) = \int_G f(x)(\gamma, -x)\,dx, \qquad \gamma \in \Gamma.$$

Plancherel's theorem states that $\mathscr{F}$ is an isometry.

If F is a Banach space and $f \in L_F^1(G)$, we can define for each $\gamma \in \Gamma$

$$\hat{f}(\gamma) = \int_G f(x)(\gamma, -x)\,dx.$$

The mapping $f \to \hat{f}$ is denoted by $\mathscr{F}$.

Now, given $F = L^r(G)$ with $1 \leqslant r < 2$, we select a function $\varphi \in L^r(G) \setminus L^2(G)$. Then the function $f\colon G \to F$ defined by $f(t) = \varphi_t$, with $\varphi_t(x) = \varphi(xt)$, belongs to $L_F^2(G)$ and has a Fourier transform

$$\hat{f}(\gamma)(\cdot) = \hat{\varphi}(\gamma)(\gamma, \cdot), \qquad \gamma \in \Gamma,$$

so $\hat{f} \notin l_F^2(\Gamma)$, since $\|\hat{f}(\gamma)\|_F = |\hat{\varphi}(\gamma)|$.

Therefore, $\mathscr{F}$ does not have a bounded extension from $L_F^2(G)$ into $l_F^2(\Gamma)$.

Indeed, for $F = l^{r'}(\Gamma)$, $2 < r' \leqslant \infty$, we take the function $g\colon G \to l^r(\Gamma)$ given by $g = \mathscr{F} \circ f$. Then

$$\hat{g}(\gamma)(\cdot) = \hat{\varphi}(\cdot)\delta_\gamma(\cdot) \quad \text{and} \quad \|\hat{g}(\gamma)(\cdot)\|_{l^{r'}(\Gamma)} = |\hat{\varphi}(\gamma)|.$$

Therefore, $\hat{g} \notin l_F^2(\Gamma)$. So the Fourier transform is not bounded from $L_F^2(G)$ into $l_F^2(\Gamma)$.

We state the following

THEOREM 4. *For a Banach space F, the following statements are equivalent:*

(a) *$\mathscr{F}$ has a bounded extension from $L_F^2(G)$ into $l_F^2(\Gamma)$.*

(b) *F is isomorphic to a Hilbert space.*

Proof. The part (b) $\Rightarrow$ (a) follows by a simple computation.

On the other hand, by the examples above and Remark 0 we see that $\mathscr{F}$ fails to have a bounded extension from $L_{l^r}^2(G)$ into $l_{l^r}^2(\Gamma)$ for $r \neq 2$. Theorem 2 states that the type of F has to be greater than r if $r < 2$, and the cotype less than r if $r > 2$; that is, F has type and cotype 2. Now Kwapień's theorem [9] gives (b).

Remarks. 1. Kwapień [9] showed that given a Banach space F, if $\mathscr{F}$ maps $L_F^2(\boldsymbol{R})$ into $L_F^2(\boldsymbol{R})$, then f is isomorphic to a Hilbert space. Therefore, standard arguments give the same statement in the case

$$\mathscr{F}\colon L_F^2(\boldsymbol{T}) \to l_F^2(\boldsymbol{Z}),$$

$\boldsymbol{T}$ being the torus.

2. By Pisier's work (see [18]) it is easy to deduce that if F is a Banach space and $\mathscr{F}$ maps $L_F^2(G)$ into $l_F^2(\Gamma)$, then F has type 2.

On the other hand, it is a trivial fact that if $\mathscr{F}$ maps $L_F^2(G)$ into $l_F^2(\Gamma)$, then it maps $L_{F^*}^2(G)$ into $l_{F^*}^2(\Gamma)$; in other words, F^* has type 2 and, consequently, F has cotype 2. Then Kwapień's theorem states that F is isomorphic to a Hilbert space.

3. Using the same argument as in Theorem 4, we can infer that if G is infinite compact and $\mathscr{F}$ maps $L_F^p(G)$ into $l_F^{p'}(\Gamma)$, then the type of F is at least p and the cotype at most p'. Some kind of converse can be obtained (see [3]).

On the other hand, it can be proved that, for a locally compact abelian group G, the natural operator which extends the Fourier transform $\mathscr{F}\colon L^p(G) \to L^{p'}(\Gamma)$, $1 < p \leqslant 2$, is the operator $\tilde{\mathscr{F}}$ mapping $L_F^p(G)$ into $V_F^{p'}(\Gamma)$, where $V_F^{p'}(\Gamma)$ is the set $\mathscr{L}(L^p(\Gamma), F)$, and $\tilde{\mathscr{F}}$ is defined by $\tilde{\mathscr{F}}f = f \circ \mathscr{F}^*$ ($\mathscr{F}^*$ denotes the adjoint of $\mathscr{F}$).

It can be shown that with the last definition $\tilde{\mathscr{F}}$ is an isometry from $V_F^2(G)$ into $V_F^2(\Gamma)$ (see [20]).

The way of proving Theorem 4 suggests a general method for which we give the following example of application:

II. It is a well-known fact that the bounded operators S from $L^2(G)$ into itself (G being an infinite compact group) that are translation invariant are exactly the operators such that

$$(Sf)\hat{\ }(\gamma) = m(\gamma)\hat{f}(\gamma), \qquad \gamma \in \Gamma,\ f \in L^2(G),$$

where $m \in l^\infty(\Gamma)$. Such sequences are called *multipliers* in $L^2(G)$ (see [11]).

Given a Banach space F it is easy to verify (considering functions $f(x) = (x, \gamma)b$, $b \in F$, $\gamma \in \Gamma$) that for a bounded translation invariant operator S from $L^2_F(G)$ into itself there exists a sequence $m(\gamma) \in l^\infty_{\mathscr{L}(F)}$ such that

$$(Sf)\hat{}(\gamma) = \langle m(\gamma), \hat{f}(\gamma)\rangle, \quad \gamma \in \Gamma,\ f \in L^2_F(G).$$

Now, to reproduce the method of Theorem 4 take $F = L^r(G)$, $1 \leqslant r \leqslant \infty$, with $r \neq 2$. It is well known that there exists a sequence $m \in l^\infty(\Gamma)$ such that the operator defined in $L^2 \cap L^r(G)$ by

$$(Sf)\hat{}(\gamma) = m(\gamma)\hat{f}(\gamma), \quad \gamma \in \Gamma,$$

does not have a bounded extension to $L^r(G)$ (see [11]).

For a trigonometric polynomial

$$\varphi(x) = \sum_{\gamma \in J} c_\gamma (\gamma, x),$$

$c_\gamma \in C$, J a finite subset of Γ, consider the $L^r(G)$-valued functions

$$f(t)(x) = \sum_{\gamma \in J} c_\gamma(\gamma, xt), \qquad g(t)(x) = \sum_{\gamma \in J} m(\gamma) c_\gamma (\gamma, xt)$$

(i.e. $f(t) = \varphi_t$ and $g(t) = (S\varphi)_t$). It is obvious that

$$\|f\|_{L^2_F(G)} = \|\varphi\|_{L^r(G)} \quad \text{and} \quad \|g\|_{L^2_F(G)} = \|S\varphi\|_{L^r(G)}.$$

Now, if $\{m(\gamma)I_F\}\, l^\infty_{\mathscr{L}(F)}(\Gamma)$ were a multiplier in $L^2_F(G)$, we would have

$$\|g\|_{L^2_F(G)} \leqslant C\|f\|_{L^2_F(G)},$$

and then $\|S\varphi\|_r \leqslant C\|\varphi\|_r$, which is a contradiction.

In other words, for each $r \neq 2$ there exists a sequence $m \in l^\infty(\Gamma)$ (an operator $S \in \mathscr{L}(L^2)$) such that the sequence $mI_{L^r} \in l^\infty_{\mathscr{L}(L^r)}(\Gamma)$ is not a multiplier for $L^2_{L^r}(G)$ (the operator $S \notin \mathscr{L}_{L^r}(L^2(G))$).

This suggests the following theorem whose proof goes as in Theorem 4 and we omit it.

THEOREM 5. *For a Banach space F the following statements are equivalent*:

(a) *The bounded translation invariant operators from $L^2_F(G)$ into itself are exactly the operators given by*

$$(Tf)\hat{}(\gamma) = \langle m(\gamma), \hat{f}(\gamma)\rangle, \quad \gamma \in \Gamma,$$

for m any sequence in $l^\infty_{\mathscr{L}(F)}(\Gamma)$.

(b) *F is isomorphic to a Hilbert space.*

5. Singular integral operators. For some operators, the property of having a bounded extension to L^p_E depends on the Banach space, but not on p. Let T be the singular integral operator in $\mathbf{R}^n$ defined by convolution with the distribution p.v. $(\Omega(x)|x|^{-n})$, where Ω is homogeneous of degree zero, has mean value zero on the unit sphere and verifies the integral Dini condition (see [21]).

THEOREM 6. *For a Banach space E, the following statements are equivalent*:

(a) T_E *is bounded in* $L^p_E(\boldsymbol{R}^n)$ *for all* p, $1 < p < \infty$.

(b) T_E *is bounded on* $L^p_E(\boldsymbol{R}^n)$ *for some* p, $1 < p < \infty$.

(c) T_E *satisfies an inequality of weak type* (1, 1).

(d) T_E *is bounded from* $L^1_E(\boldsymbol{R}^n)$ *to* $L^0_E(\boldsymbol{R}^n)$ (*with the topology of local convergence in measure*).

Proof. A repetition of the usual argument for scalarly-valued functions shows that (b) implies (c), and on the other hand, from (b) and (c) together one obtains (a) by interpolation and duality (see [1] for details). Thus, it only remains to prove that (d) implies (b). Suppose that (d) holds. This implies first of all that $p(E) > 1$, since T is not regular (i.e., it is not dominated by a positive continuous operator). Take $1 \leqslant p < p(E)$ and $v(x) > 0$ such that

$$v \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n) \quad \text{and} \quad v^{-1} \in L^{1/(p-1)}(\boldsymbol{R}^n).$$

Then $L^p_E(v) = L^p_E(\boldsymbol{R}^n, v(x)\,dx)$ imbeds continuously into $L^1_E(\boldsymbol{R}^n)$, and T_E is well defined on $L^p_E(v)$ and continuous in measure. Since $L^p_E(v)$ is of type p, it follows from Nikishim's theorem (see [13]) that there exists $u(x) > 0$, $u \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$, such that

$$\int_{\{\|T_E f(x)\|_E > \lambda\}} u(x)\,dx \leqslant \lambda^{-p} \int \|f(x)\|^p_E\, v(x)\,dx, \qquad f \in L^p_E(v). \tag{5}$$

By using the translation and dilation invariance of T, we can replace $u(x)$ and $v(x)$ in (5) by $u(\delta x + h)$ and $v(\delta x + h)$, respectively, for arbitrary $\delta > 0$ and $h \in \boldsymbol{R}^n$. Then we integrate with respect to h over the unit ball

$$B = \{h \in \boldsymbol{R}^n \colon |h| \leqslant 1\}$$

and let $\delta \to 0$ to obtain

$$|\{x \colon \|T_E f(x)\|_E > \lambda\}| \leqslant C_p \lambda^{-p} \int \|f(x)\|^p_E\, dx. \tag{6}$$

Inequality (6) holds for all p $(1 \leqslant p \leqslant p(E))$, and interpolation gives (b).

Theorem 4 was essentially known, but the form given here is somewhat sharper. A different proof has been given by Virot [24] in the case of the Hilbert transform, based on "good λ inequalities". The proof presented here is different in spirit and essentially taken from [22]. It can be applied to operators for which no "good λ inequality" is available.

Burkholder, Bourgain and McConnell have obtained the exact geometrical condition for a Banach space E in order to verify some of the conditions in Theorem 6.

The geometrical condition is the existence of a real function ζ on $E \times E$ having the following properties:

(i) $\zeta(x, \cdot)$ is convex for each $x \in E$;

(ii) $\zeta(x, y) = \zeta(y, x)$;

(iii) $\zeta(x, y) \leqslant \|x+y\|_E$ if $\|x\|_E \leqslant 1 \leqslant \|y\|_E$;

(iv) $\zeta(0, 0) > 0$.

This condition was introduced by Burkholder [5], and it is called ζ-*convexity*.

6. Dyadic decomposition. Let Δ be the dyadic decomposition of $\boldsymbol{R}$, i.e., the family of intervals $[-2^{k+1}, -2^k]$, $[2^k, 2^{k+1}]$, $-\infty < k < \infty$.

Let f be an E-valued function, and I be a dyadic interval. The function $S_I f$ such that

$$(S_I f)\hat{}(\xi) = \chi_I(\xi)\hat{f}(\xi), \quad \xi \in \boldsymbol{R},$$

is well defined for good f.

Very recently McConnell [15] has proved the following multiplier theorem:

THEOREM 7. *Let E be a ζ-convex Banach space and m a C^2-function on $\boldsymbol{R}$. Assume there is a constant c_m such that*

$$|\xi|^\alpha \left|\frac{d^\alpha m(\xi)}{d\xi^\alpha}\right| \leqslant c_m, \quad \alpha = 0, 1, 2.$$

Then the operator T_m defined, for good functions, as the function $T_m f$ such that

$$(T_m f)\hat{}(\xi) = m(\xi)\hat{f}(\xi), \quad \xi \in \boldsymbol{R},$$

has an extension to $L_E^p(\boldsymbol{R})$, $1 < p < \infty$, which satisfies

$$\|T_m f\|_{L_E^p} \leqslant c_p \|f\|_{L_E^p}.$$

In the case where E is a Banach lattice one can consider the Littlewood–Paley operator

$$Sf(x) = \Big(\sum_{I \in \Delta} |S_I f(x)|^2\Big)^{1/2},$$

where $|\cdot|$ is the absolute value in E. Then we have the following

THEOREM 8. *Let E be a Banach lattice. The following conditions are equivalent*:

(a) *There exists a constant c_p such that*

$$c_p^{-1}\|f\|_{L_E^p} \leqslant \|Sf\|_{L_E^p} \leqslant c_p \|f\|_{L_E^p}.$$

(b) *E is ζ-convex.*

Proof. (a) ⇒ (b) is obvious, since by the properties of lattices we have

$$S_I f(x) \leqslant Sf(x) \quad (x \in \boldsymbol{R},\ I \text{ dyadic})$$

and

$$\|S_I f\|_{L^p_E} \leqslant c_p \|f\|_{L^p_E}.$$

Conversely, let φ be a Schwartz function such that

$$\chi_{[1,2]} \leqslant \varphi \leqslant \chi_{[2^{-1},4]}.$$

Write $\varphi_{2^k}(x) = \varphi(2^{-k}x)$, $k \in \boldsymbol{Z}$. It is a well-known fact that if $T_k f$ is the operator defined by

$$(T_k f)\hat{}(\xi) = \varphi_{2^k}(\xi)\hat{f}(\xi), \qquad \xi \in \boldsymbol{R},$$

and $I_k = [2^k, 2^{k+1}]$, then

$$S_{I_k} T_k f = S_{I_k} f.$$

Now, as usual in the Littlewood–Paley decomposition (see [21]), we consider for each $t \in [0, 1]$ the multiplier

$$m_t(\xi) = \sum_{k \in \boldsymbol{Z}} r_k(t)\varphi_{2^k}(\xi), \qquad \xi \in \boldsymbol{R},$$

r_k being the Rademacher functions.

By the choice of φ we see that m_t satisfies the hypothesis of McConnell's theorem, and then

$$\|T_{m_t} f\|_{L^p_E} \leqslant c_p \|f\|_{L^p_E}, \qquad t \in [0, 1],$$

with c_p independent of t. Hence

$$\int_0^1 \|T_{m_t} f\|_{L^p_E}\, dt \leqslant c_p \|f\|_{L^p_E}, \qquad 1 < p < \infty.$$

Now E is a ζ-convex Banach lattice, and then the same is true for L^p_E. Then L^p_E is of type r for some $r > 1$, and therefore (see [12]) the Khintchine's inequality for the absolute value is valid. This means that

$$\|(\sum_k |T_k f|^2)^{1/2}\|_{L^p_E} \leqslant C \int_0^1 \|\sum_k r_k(t) T_k f\|_{L^p_E}\, dt \leqslant C'_p \|f\|_{L^p_E}.$$

On the other hand, by a direct application of Krivine's theorem (cf. [12]) we see that for E a ζ-convex Banach lattice and $1 < p < \infty$

$$\|(\sum_k |S_{I_k} f|^2)^{1/2}\|_{L^p_E} \leqslant A_p \|(\sum_k |f_k|^2)^{1/2}\|_{L^p_E}.$$

The standard truncation argument gives, for $1 < p < \infty$,

$$\|(\sum_k |S_{I_k} f|^2)^{1/2}\|_{L^p_E} \leqslant B_p \|f\|_{L^p_E}.$$

To obtain the left-hand side of the inequality take $f \in E \otimes S(\mathbf{R})$ and $g \in E^* \otimes S(\mathbf{R})$. Then

$$\sum_k \int_{\mathbf{R}} \langle S_{I_k} f(x), S_{I_k} g(x) \rangle dx = \int_{\mathbf{R}} \langle f(x), g(x) \rangle dx.$$

Therefore, by the properties of lattices (see [12]) we obtain

$$\int_{\mathbf{R}} \langle f(x), g(x) \rangle dx \leqslant \int_{\mathbf{R}} \langle (\sum_k |S_{I_k} f(x)|^2)^{1/2}, (\sum_k |S_{I_k} g(x)|^2)^{1/2} \rangle dx.$$

As E is ζ-convex, we have

$$\|f\|_{L^p_E} \leqslant C_p \|(\sum_k |S_{I_k} f(x)|^2)^{1/2}\|_{L^p_E}.$$

Remarks. 1. If we consider the Littlewood–Paley operator

$$Gf(x) = (\sum_{I \in \Delta} \|S_I f(x)\|_E^2)^{1/2},$$

then any inequality of the form

$$A_p \|f\|_{L^p_E} \leqslant \|Gf\|_{L^p} \leqslant B_p \|f\|_{L^p_E}$$

implies that E is isomorphic to a Hilbert space. This can be seen using the general method described in Section 4, and also it comes from Pisier's work (see [18]).

2. In the case where E is a ζ-convex lattice with an unconditional basis, Theorems 7 and 8 are essentially contained in [19] since

$$Sf = \sum (Sf_i) a_i,$$

where (a_i) is the unconditional basis.

REFERENCES

[1] A. Benedek, A. P. Calderón and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U.S.A. 48 (1962), pp. 356–365.

[2] J. Bourgain, *Remarks on Banach spaces in which martingale difference sequences are unconditional* (to appear).

[3] — *A Hausdorff–Young inequality for B-convex Banach spaces*, Pacific J. Math. 101 (1982), pp. 255–262.

[4] B. V. Buhvalov, *Continuity of operators in spaces of measurable vector-valued functions with applications*, Soviet Math. Dokl. 20 (1979), pp. 480–484.

[5] D. L. Burkholder, *A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional*, Ann. Probability 9 (1981), pp. 997–1011.

[6] — *A geometrical condition that implies the existence of certain singular integrals of Banach space valued functions*, pp. 270–286 in: *Conference on Harmonic Analysis in Honor of A. Zygmund*, W. Beckner, A. P. Calderón, R. Fefferman and P. W. Jones editors, Wadswoth, Belmont, CA, 1983.

[7] C. Herz, *The theory of p-spaces with an application to convolution operators*, Trans. Amer. Math. Soc. 154 (1971), pp. 69–82.

[8] R. C. James, *Nonreflexive space of type 2*, Israel J. Math. 30 (1978), pp. 1–13.

[9] S. Kwapień, *Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients*, Studia Math. 44 (1972), pp. 583–595.

[10] — *Operator factorizable through L^p-spaces*, Bull. Soc. Math. France Mém. 31–32 (1972), pp. 215–222.

[11] R. Larsen, *An Introduction to the Theory of Multipliers*, Springer-Verlag, 1971.

[12] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. II*, Springer Ergebnisse der Math. 97 (1979).

[13] B. Maurey, *Nouveaux théorèmes de Nikishin*, Séminaire Maurey–Schwartz, 1973–1974, Ecole Polytechnique, Palaiseau.

[14] — and G. Pisier, *Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach*, Studia Math. 58 (1976), pp. 45–90.

[15] T. R. McConnell, *On Fourier multiplier transformations of Banach valued functions* (to appear).

[16] G. Pisier, *Sur les espaces de Banach qui ne contiennent pas uniformément de l_n^1*, C. R. Acad. Sci. Paris, Série A, 277 (1973), pp. 991–994.

[17] — *Semi-groupes holomorphes et K-convexité*, Séminaire d'Analyse Fonctionnelle 1980–1981, Ecole Polytechnique, Palaiseau.

[18] — *Type des espaces normés*, Séminaire Maurey–Schwartz 1973–1974, Ecole Polytechnique, Palaiseau.

[19] J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea, *Calderón–Zygmund theory for operator valued kernels* (to appear).

[20] F. J. Ruiz and J. L. Torrea, *Transformada de Fourier de funciones vectoriales*, Rev. Real Acad. Ci. Exact. Fís. Natur. Madrid 75 (1981), pp. 707–717.

[21] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, N. J., 1970.

[22] J. L. Torrea, *Analisis de Fourier de funciones vectoriales*, Doctoral dissertation, Universidad de Zaragoza, 1980.

[23] B. Virot, *Extensions vectorielles d'opérateurs linéaires bornés sur L^p*, C. R. Acad. Sci. Paris, Série A, 293 (1981), pp. 413–415.

[24] — *Quelques inégalités concernant les transformes de Hilbert de fonctions à valeurs vectorielles*, ibidem 293 (1981), pp. 413–415.

[25] A. Zygmund, *Trigonometric Series*, Cambridge 1959.

DIVISION DE MATEMATICAS
UNIVERSIDAD AUTONOMA DE MADRID
SPAIN

Reçu par la Rédaction le 1. 9. 1984