COLLOQUIUM MATHEMATICUM

VOL. XIII 1965 FASC. 2

CURVE BREAKERS

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1. Introduction. The concept of connectedness for topological spaces generates among many other concepts the concept of cut point. The point p of the connected topological space S is a cut point in S if and only if $S-\{p\}$ is not connected. The purpose of this note is to introduce a new topological concept—a curve breaker—which is the analogue for pathwise connected spaces of the cut point for connected spaces. All curve breakers are non-cut points but not all non-cut points are curve breakers.

The theorems in section 2 are direct analogues of theorems in a book of Wilder [1] (chapter I, section 10).

Throughout this note a *curve* in a space S will mean, as usual, the range in S of a continuous mapping of the subspace, $\{x|\, 0 \leqslant x \leqslant 1\}$, of the usual space of real numbers, i. e. the unit interval space $[0,1]$ into S. Any continuous mapping of the unit interval space $[0,1]$ will be called a *path* in S. A space is, then, pathwise connected if and only if for each two points a and b in S there exists at least one path f in S such that $f(0) = a$ and $f(1) = b$.

DEFINITION 1.1. A point p in a pathwise connected space S is called a *curve breaker* in S if and only if $S-\{p\}$ is connected but not pathwise connected.

DEFINITION 1.2. A point p in a pathwise connected space S is called a *curve leaver* in S if and only if $S-\{p\}$ is pathwise connected.

Consequently, the points in any pathwise connected space fall into three disjoint classes: (1) cut points, (2) curve breakers and (3) curve leavers.

2. Properties of curve breakers. The following definitions are in the book of Wilder for T_1-spaces. A connected space S is called *irreducibly connected* about a subset M if and only if no proper connected

[1] R. L. Wilder, *Topology of manifolds*, American Mathematical Society Colloquium Publication 32, New York 1949.

subset of S contains M. A connected space S has a basic set B about which it is irreducibly connected if and only if S is irreducibly connected about B but is not irreducibly connected about any proper subset of B. Wilder established that a space S is irreducibly connected about a subset M if and only if x in $S-M$ implies that x is a cut point and any separation $A \cup B$ of $S-\{x\}$ implies that $M \cap A \neq \emptyset \neq M \cap B$. In the book of Wilder it is also proved that 1° a space S is its own basic set if and only if S consists entirely of non-cut points, and 2° a space S has a basic set about which it is irreducibly connected if and only if S is irreducibly connected about its set of non-cut points.

The analogues of these definitions and theorems will be established for curve breakers in T_1-spaces.

DEFINITION 2.1. A pathwise connected space S is said to be *irreducibly pathwise connected* about a subset M if and only if no proper pathwise connected subset of S contains M.

DEFINITION 2.2. A pathwise connected space S is said to have a *basic set* B about which it is irreducibly pathwise connected if and only if S is irreducibly pathwise connected about B but is not irreducibly pathwise connected about any proper subset of B.

THEOREM 2.3. *A pathwise connected space S is irreducibly pathwise connected about a subset M of S if and only if M contains all curve leavers of S and M intersects, on a non-null set, at least two path components of $S-\{x\}$ for every x in $S-M$.*

Proof. Let S be irreducibly pathwise connected about M. Let x be in $S-M$. $S-\{x\}$ contains M and is a proper subset of S. Hence, $S-\{x\}$ is not pathwise connected. x is then a curve breaker or a cut point and M must contain all curve leavers of S. Further, if M lay in one path component of $S-\{x\}$, then S would not be irreducibly pathwise connected about M.

Let M contain all of the curve leavers in the pathwise connected space S and, for every x in $S-M$, let M intersect at least two path components of $S-\{x\}$. Now, let A be a proper pathwise connected subset of S which contains M. Let x be in $S-A$. x is, then, in $S-M$ and so is a cut point or a curve breaker and $S-\{x\}$ is not pathwise connected. $S-\{x\} \supseteq A$ and M lies in several path components of $S-\{x\}$ by hypothesis. Hence, A lies in several path components of $S-\{x\}$ and this is a contradiction. Thus S is irreducibly pathwise connected about M.

COROLLARY 2.4. *If S is irreducibly pathwise connected about a set M, then M contains the set L of curve leavers of S.*

COROLLARY 2.5. *If S is irreducibly pathwise connected about the set L of curve leavers of S, then L is a basic set about which S is irreducibly pathwise connected.*

LEMMA 2.6. *Let x be a point of a pathwise connected T_1-space S. If C is a path component of $S-\{x\}$, then $C \cup \{x\}$ is pathwise connected.*

Proof. Since S is pathwise connected, for any point y in C there is a curve γ in S containing x and y. Let λ be a continuous function whose range is γ and let $\lambda(0) = y$ and $\lambda(1) = x$. Since S is a T_1-space, $\{x\}$ is closed in S and $\lambda^{-1}(x)$ is closed in $[0, 1]$. Further, $\lambda^{-1}(x) \subseteq \sim\{0\}$, the complement of $\{0\}$. $\lambda^{-1}(x)$ is closed and bounded and so contains its greatest lower bound a. Since $\lambda(a) = x$, $a \neq 0$. Hence, the neighborhood $U_a = \{t | 0 \leqslant t < a\}$ is mapped by λ into $S-\{x\}$. If s is in U_a and if $\lambda(s)$ is in a path component of $S-\{x\}$, other than C, then there exists a curve from $\lambda(s)$ to y in $S-\{x\}$ and C is not a path component. This is a contradiction and so $\lambda[U_a] \subseteq C$ and $\lambda[U_a \cup \{a\}] \subseteq C \cup \{x\}$. Thus $C \cup \{x\}$ is pathwise connected.

THEOREM 2.7. *If a pathwise connected T_1-space S has a basic set B about which S is irreducibly pathwise connected then B is the set of curve leavers of S.*

Proof. Let S be irreducibly pathwise connected about the subset B as basic set. By corollary 2.4, $B \supseteq L$ where L denotes the set of curve leavers of S. Let x be in B. Since S is irreducibly pathwise connected about B as basic set, S is not irreducibly pathwise connected about $B-\{x\}$, by definition 2.2. Hence, there exists a proper pathwise connected subset A of S which contains $B-\{x\}$. x is not in A; for, then, A would contain B and S would not be irreducibly pathwise connected about B. Let C_A denote the path component of $S-\{x\}$ which contains A. By lemma 2.6, $C_A \cup \{x\}$ is pathwise connected and, by definition, $C_A \cup \{x\}$ contains B. Hence, $C_A \cup \{x\} = S$. This means that $S-\{x\} = C_A$ and so x is in L.

COROLLARY 2.8. *A pathwise connected T_1-space can have at most one basic set about which it is irreducibly pathwise connected.*

COROLLARY 2.9. *A necessary and sufficient condition that a pathwise connected T_1-space be irreducibly pathwise connected about itself as basic set is that it consist entirely of curve leavers.*

COROLLARY 2.10. *A necessary and sufficient condition that a pathwise connected T_1-space S have a basic set about which it is irreducibly pathwise connected is that S be irreducibly pathwise connected about its set of curve leavers.*

3. Examples of curve breakers. Example 3.1. Let $\mathscr{S}$ be the subspace of the plane E^2 shown in fig. 1.

$\mathscr{S}$ is the union of a countable chained system of line segments $[\alpha_1, \beta_1]$, $[\alpha_2, \beta_2]$, ..., $[\alpha_n, \beta_n]$, ... where each $[\alpha_i, \beta_i]$ for $i = 1, 2, \ldots$ is the homeomorphic image of an interval $[a_i, b_i]$ of real numbers. Further, since

$[\alpha_i, \beta_i] \cap [\alpha_j, \beta_j] = \emptyset$ for $i+1 < j$ and $[\alpha_i, \beta_i] \cap [\alpha_{i+1}, \beta_{i+1}] = \beta_i = \alpha_{i+1}$, $\mathscr{S}$ can be considered to be the continuous image of the subspace of E_1 consisting of the non-negative real numbers under a 1-1 mapping ψ which is locally homeomorphic except at 0, where $\psi(0) = (0, 0)$, $\psi(1) = (-1, 0)$, $\psi(2) = (-1, -2)$, etc.

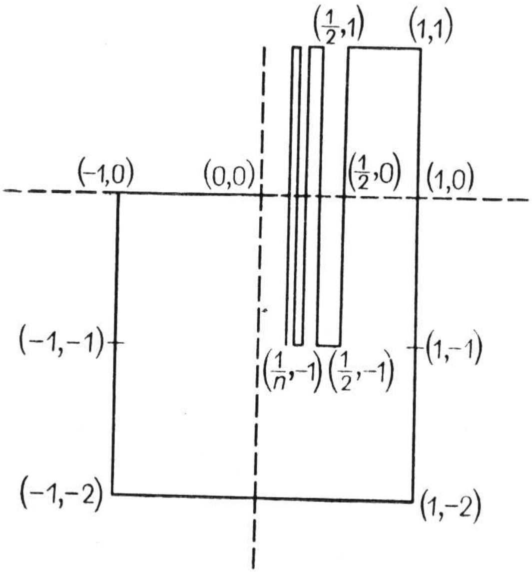

Fig. 1

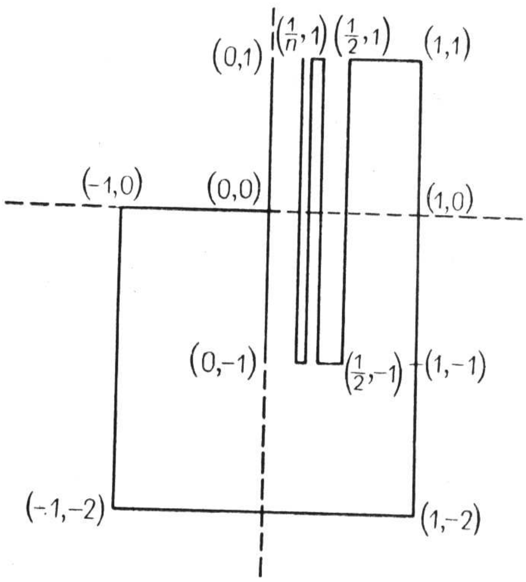

Fig. 2

$\mathscr{S}$ is pathwise connected and has no cut points. If (a, b) is in $\mathscr{S}$ and $(a, b) \neq (0, 0)$, then (a, b) is a curve breaker in $\mathscr{S}$. $(0, 0)$ is a curve leaver in $\mathscr{S}$.

$\mathscr{S}$ is irreducibly connected about itself as basic set but $\mathscr{S}$ has no basic set about which it is irreducibly pathwise connected. Indeed, $\mathscr{S}$ has no cut points. Hence by corollary 10.20 in chapter I of the book of Wilder, $\mathscr{S}$ is its own basic set about which it is irreducibly connected. By theorem 2.7, if $\mathscr{S}$ has a basic set about which it is irreducibly pathwise connected, then this set must be the set of curve leavers of $\mathscr{S}$. However, the set of curve leavers of $\mathscr{S}$ is $\{(0, 0)\}$ and $\mathscr{S}$ is not irreducibly pathwise connected about $\{(0, 0)\}$.

Example 3.2. This example describes a space which is compact, pathwise connected and separable but which is not irreducibly pathwise connected about its set of curve leavers. Thus there is no direct analogue for theorem 10.30 in chapter I of the book of Wilder for curve leavers. The desired space $\mathscr{T}$ is sketched in fig. 2. It is a subspace of the plane.

$\mathscr{T}$ is the union of the space $\mathscr{S}$ of example 3.1 and the subspace D of E^2 consisting of $\{(x, y) | x = 0 \text{ and } -1 \leqslant y \leqslant 1\}$. Since $\mathscr{T}$ is a closed bounded subset of E^2, $\mathscr{T}$ is compact and separable. Further, since $\mathscr{T}$ is the union of two connected sets which intersect on $\{(0, 0)\}$, $\mathscr{T}$ is connected.

$\mathscr{T}$ is pathwise connected and has two curve leavers; namely, $(0, -1)$ and $(0, 1)$. The other points of $\mathscr{T}$ are curve breakers. Thus $\mathscr{T}$ has no cut points.

Example 3.3. Let $\mathscr{U}$ be the subspace of E^2 described in fig. 3.

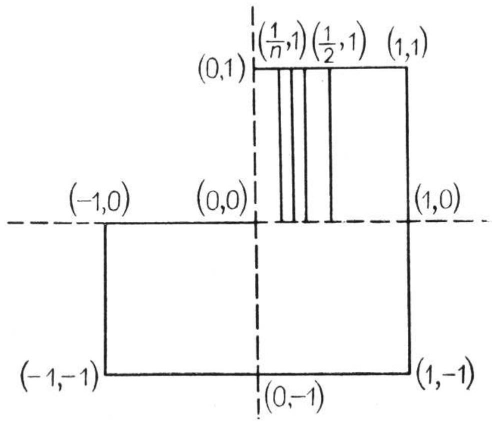

Fig. 3

$\mathscr{U}$ is pathwise connected. The points $(1/n, y)$ for $n = 1, 2, \ldots$ and $0 < y \leqslant 1$ are cut points of $\mathscr{U}$. The points $(0, 0)$, $(0, 1)$ and $(1/n, 0)$ for $n = 1, 2, \ldots$ are curve leavers. The rest of the points in $\mathscr{U}$ are curve breakers. Thus $\mathscr{U}$ has an infinite number of the three types of points which can occur in a pathwise connected space.

Reçu par la Rédaction le 31. 8. 1964;
en version modifiée le 24. 11. 1964