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CERTAIN QUESTIONS RELATED TO THE EQUIVALENCE OF LOCAL CONNECTEDNESS AND CONNECTEDNESS IM KLEINEN

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1. Introduction. For each of the three properties, (1) local connectedness, (2) aposyndesis, and (3) semi-local-connectedness, of a compact metric space, there can be defined a corresponding property (connectedness im kleinen in the first case) which differs from the given property only in the requirement that a certain set be open instead of closed or closed instead of open. In view of the equivalence of local connectedness to connectedness im kleinen, it is interesting to consider the question of equivalence of the others to their corresponding properties.

This paper * gives a single example of a bounded, plane continuum showing (1) that the only two of the corresponding properties that are equivalent globally are local connectedness and connectedness im kleinen, (2) that in none of these cases is the total absence of one of the properties equivalent to the total absence of the corresponding property, and (3) that, unlike total non-aposyndesis and total non-semi-local-connectedness, total absence of the two corresponding new properties, in compact metric spaces, does not imply the existence of weak cut points.

Other examples are given showing that neither total non-aposyndesis nor total non-semi-local-connectedness implies the other for bounded plane continua. The question of equivalence of these two arises naturally from the equivalence of aposyndesis and semi-local-connectedness for compact metric spaces and, in addition, is of interest in connection with weak cut point theory ([1], Theorem 4).

2. Definitions. The following definitions are stated in a way that emphasizes the relationship between corresponding properties.

A connected topological space is *locally connected* (respectively, *connected im kleinen*) at a point p if each open set D containing p contains

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an open (resp., closed) subset (relative to D) that is a connected neighborhood of p.

A connected topological space M is *domain aposyndetic* (respectively, *aposyndetic*) at a point p if, for each distinct point q, there is a neighborhood D_p of p, a neighborhood D_q of q, and an open (resp. closed) connected set A such that $M-D_q \supset A \supset D_p$.

A connected topological space is *semi-connected im kleinen* (respectively, *semi-locally-connected*) at a point p if each open set containing p contains a closed (resp., open) neighborhood of p, the complement of which consists of a finite number of components.

3. Examples. The following example indicates the extreme extent to which all pairs of these properties fail to be equivalent as point properties and shows that local connectedness and connectedness im kleinen are the only ones that even come close to being equivalent globally.

It is obvious that one of the properties in each pair implies its corresponding property as a point property.

Example 1. An aposyndetic semi-locally-connected bounded, plane continuum H which is connected im kleinen on a dense G_δ set and has no non-dense connected open subset; and, therefore, is not (1) domain aposyndetic, (2) semi-connected im kleinen, or (3) locally connected, at any point.

The desired continuum H is described as the common part of a sequence $H_0, H_1, \ldots$ of plane continua defined by induction.

Let D_0 be the join of the two points $P = (\frac{1}{2}, \frac{1}{2})$ and $Q = (\frac{1}{2}, -\frac{1}{2})$ with the unit interval. Let K_0 be the join of P and Q with the Cantor set (on the unit interval). Further, let the complementary domains of K_0, in D_0, be divided into three classes A, B and C such that the closure of the union of each class contains K_0. Let $H_0 = D_0$ and $H_1 = D_0 - C^*$ (i. e., H_1 is the set of all points in D_0 which are not in any member of C).

The closure of each member T of A and B is triangulated in such a way that (1) two sides of each triangle in $T-(P \cup Q)$ separate P from Q in T, (2) no four of the triangles in T intersect, (3) no triangle is of diameter greater than $1/2$, and (4) all but a finite number of the triangles thus gotten in H_1 are as small as one wishes. These triangular disks are considered to be the *distinguished subdisks* of H_1. The points P and Q are the *distinguished points* of D_0, and D_0 is the distinguished subdisk of H_0.

If T is in A (respectively, T is in B) and abc is a triangle in the triangulation of T such that ab separates c from P (resp. from Q) in T, then a and b are the distinguished points of abc.

Assume H_i, its distinguished subdisks, and their distinguished points are defined. Let $D_1, D_2, \ldots$ be a counting of the distinguished subdisks of H_i and, for each j, let P_j and Q_j be the distinguished points of D_j. Also, for each j, let h_j be a homeomorphism of D_0 onto D_i such that $h_j(P) = P_j$ and $h_j(Q) = Q_j$. Let H_{i+1} be the closure of $\bigcup_{j=1}^{\infty} h_j(H_1)$. In addition let the distinguished subdisks of H_{i+1}, and their distinguished points, be the images under the h_j of the distinguished subdisks of H_1 and their distinguished points. The h_j are chosen in such a way that all but a finite number of the distinguished subdisks of H_{i+1} are as small as one may wish and also in such a way that no distinguished subdisk of H_{i+1} has diameter greater than $(1/2)^{i+1}$.

Let $H = \bigcap_{i=0}^{\infty} H_i$.

To see that H is aposyndetic at any point p of H, with respect to any other point q of H, let j be the least natural number i such that p and q do not belong to the same distinguished subdisk of H_i, and consider the following main cases.

Case 1. $j = 1$.

Case 2. $j > 1$. Let D be the distinguished subdisk of H_{j-2} to which p and q belong.

(i) The point p is a boundary point of D; in which case p is a distinguished point of D and q is an interior point of D.

(ii) The point p is an interior point of D.

In Case 2, one need only show that there is a subcontinuum K of $D \cap H - q$ that contains p in its interior relative to $D \cap H$. It is convenient to choose K as the intersection with H of the closure of the union of a collection of distinguished subdisks of H_j contained in D. Case 1 is treated in much the same way as Case 2 (ii). The continuum called for in the definition of aposyndesis can be taken as the continuum K, chosen above, together with, in Case 2 (i), the closure of $H - D \cap H$.

Every aposyndetic, compact, metric continuum (including H) is semi-locally-connected ([2], Theorem 4).

It is clear that H is connected im kleinen at each point of the dense G_δ set consisting of the common part of the interiors, in the plane, of $H_1, H_2, \ldots$

The construction is such that each connected open subset of H_{i+1} that contains a distinguished point of some distinguished subdisk D of H_i must also contain a distinguished point of each distinguished subdisk of H_i contained in the same distinguished subdisk D' of H_{i-1} as D, and also must contain a distinguished point of D'. This, together with the fact that any set of points containing a distinguished point from each distinguished subdisk of H_i is $((1/2)^{i-1})$-dense in H, for each i, implies that each connected open subset of H is dense in H. This in turn implies that H is neither domain aposyndetic, semi-connected im kleinen, nor locally connected at any point.

Although H is totally non-domain aposyndetic and totally non-semi-connected im kleinen, it contains no weak cut points [3].

Example 2. A totally non-semi-locally-connected, bounded, plane continuum that is connected im kleinen, and hence aposyndetic, at each point of a dense G_δ set.

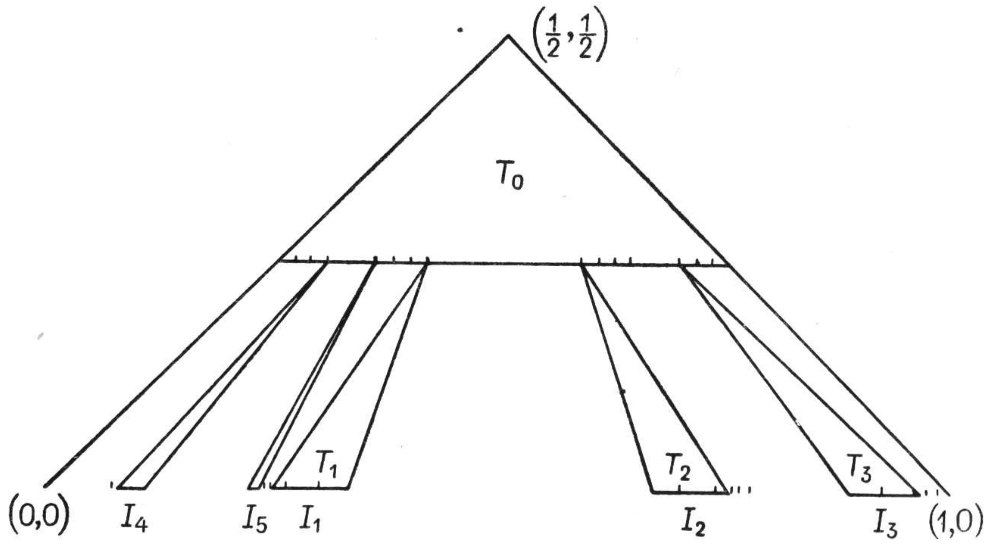

Fig. 1

The desired continuum K is described in terms of the following construction of a continuum K'. The construction of K' is similar to that for Example 1.

Let D_0 be the join of the unit interval I and the point $P = (\frac{1}{2}, \frac{1}{2})$. Let C_0 be the Cantor set (on I). Let I' be an upper semi-continuous decomposition of I into points and intervals such that (1) each non-degenerate element of I' has its end points in C_0 and intersects C_0 in a perfect set, and (2) the closure, in I', of the set of non-degenerate elements of I' is a Cantor set containing all the degenerate elements of I' that are in C_0. Let h be a homeomorphism of the arc I' onto $I_0 = \{(x, y) \text{ in } D_0 | y = \frac{1}{4}\}$ that preserves order from left to right. Let $I_1, I_2, \ldots$ be a counting of the non-degenerate elements of I'. Let T_0 be the triangular disk with vertex P and base I_0. Let P be the *distinguished point* of T_0 and let the closure of $\bigcup_{i=1}^{\infty} h(I_i)$ be the *distinguished Cantor set* of T_0. For each $i > 0$, let T_i be the triangular disk with vertex $h(I_i)$ and base I_i. The point $h(I_i)$ is the distinguished point and $I_i \cap C_0$

is the distinguished Cantor set of T_i. Let K_1' be the closure of $\bigcup_{i=0}^{\infty} T_i$, as indicated in Fig. 1.

The construction proceeds by induction, as in the preceding example, using homeomorphisms of D_0 onto the distinguished subdisks of K_i' which carry C_0 onto distinguished Cantor sets and P onto distinguished points. The homeomorphisms are chosen so that all the distinguished subdisks of K_i' are of diameter not greater than $1/2^i$.

The continuum $K' = \bigcap_{i=1}^{\infty} K_i'$ is connected im kleinen at each point of a dense G_δ set (the common part of the interiors, in the plane, of $K_1', K_2', \ldots$), and is non-semi-locally-connected at each point of $K' - C_0$. However, it is semi-locally-connected at each point of C_0. This defect can be corrected by the following change in the construction. Let $K_1 = K_1'$ (with the same distinguished subsets). To get K_{i+1} from K_i use homeomorphisms of D_0 onto the distinguished subdisks of K_i as before, except in the case of distinguished subdisks T of K_i containing points of C_0, i. e., having their bases on I. For each of these disks use a homeomorphism of D_0 *into* the disk as indicated in Fig. 2. The homeomorphisms on D_0 are chosen so that each distinguished subdisk of K_{i+1} is of diameter not greater than $1/2^{i-1}$, except those having their distinguished Cantor set on I: these all being of diameter greater than some positive number ε (independent of i), and being such that any monotone sequence of them has an arc as common part. Furthermore, the homeomorphisms are chosen so that any irreducible subcontinuum in $K = \bigcap_{i=1}^{\infty} K_i$, from a point in the common part of a monotone sequence of distinguished disks intersecting C_0 to a point not in the common part, contains a $(\sin x^{-1})$-continuum having the common part as limiting interval. The bounded plane continuum K has the required properties.

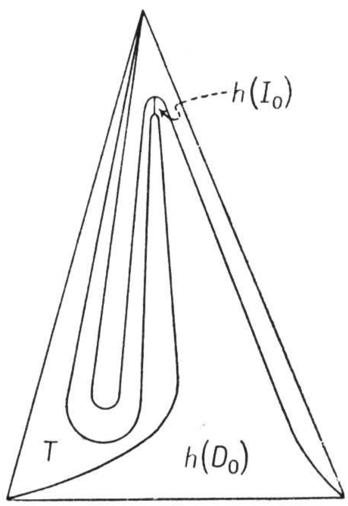

Fig. 2

A simpler example K'' of a totally non-semi-locally-connected, compact, metric continuum that is connected im kleinen on a dense G_δ subset, exists in E^3. Let $f(x, y) = \sin y^{-1}$, for (x, y) in K' and $y > 0$. Let K'' be the closure of the graph of f.

In a sense, K is as aposyndetic as a totally non-semi-locally-connected, compact, metric continuum can be, since a compact metric continuum is semi-locally-connected at each point of any open set on which it is aposyndetic ([2], Theorem 4 (Proof)).

Example 3. A totally non-aposyndetic, bounded, plane continuum that is semi-locally-connected at all but two points.

Let X and Y be two Cantor fans of circles with vertices x and y, respectively. Further, let I_x and I_y be arcs in X and Y, each having the appropriate vertex as end point. Let h be a homeomorphism of I_x onto I_y such that $h(x) \neq y$. Let M be the union of X and Y, identifying each z of I_x with $h(z)$ in I_y. Then M is a totally non-aposyndetic compact continuum that is semi-locally-connected at each point other than x and y and which can be imbedded in the plane.

M is as semi-locally-connected as a totally non-aposyndetic continuum can be.

THEOREM. *If T is a totally non-aposyndetic, connected topological space, then T is non-semi-locally-connected at some two points.*

Proof. If T is semi-locally-connected at a point q, then M is aposyndetic at each point p of $T-q$ with respect to q ([2], Theorem 3 (Proof)). Since T is non-aposyndetic, T contains a point q at which it is non-semi-locally-connected. Since T is totally non-aposyndetic, T is non-aposyndetic at q with respect to some point p. Hence T is non-semi-locally-connected at p as well as at q.

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