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CONCERNING THE CHARACTERIZATION OF LINEAR SPACES

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The reader should not be misled as to the importance of this paper by the number of its authors. It is simply a result of their common talks.

In a torsion free Abelian divisible group G there is a uniquely determined multiplication of elements by rational numbers. By a *metric group* we understand here a group with an *invariant* metric, i. e. with a metric $\varrho(x, y)$ such that $\varrho(x, y) = \varrho(x+a, y+a)$. We write $\|x\| = \varrho(x, 0)$. If G is metric, complete and separable, then $x_n \to 0$ implies $wx_n \to 0$ for every rational w. It is obviously sufficient to show this for $w = 1/m$ (m integer) and in this case it follows by putting $H = G$ and $\varphi(x) = mx$ from Banach's theorem on the continuity of inverse operation φ^{-1}, where φ is a continuous isomorphism of G onto a metric and complete group H (see [1]). If G is not separable, then $x_n \to 0$ does not imply $wx_n \to 0$. However, the following theorem will be proved in the sequel:

THEOREM 1. *If G is a metric and complete torsion free Abelian divisible group and if $\lim_n (y/n) = 0$ for every $y \in G$, then $x_n \to 0$ implies $wx_n \to 0$ for every rational w.*

Thus a group fulfilling the assumption of Theorem 1 becomes a metric linear space over rationals if and only if the following condition is satisfied:

(*) $w_n \to 0$ implies $w_n x \to 0$ for every x.

In this paper we are concerned first of all with the following problem:

(Q) Is it possible to drop condition (*) in the characterization of a linear space, i. e., does (*) follow already from the assumptions of Theorem 1?

The answer is "yes" for locally compact groups but "no" in general, as is shown by following theorems:

THEOREM 2. *If G is a locally compact (may be non-metrizable) torsion free Abelian divisible group, then, for every $x \in G$, $\lim_n (x/n) = 0$ implies $w_n x \to 0$ whatever be a sequence of rationals $w_n \to 0$.*

THEOREM 3. *There is a metric complete separable torsion free Abelian divisible group such that $\lim_n (x/n) = 0$ for every x but $w_n x \not\to 0$ for some x and some sequence $w_n \to 0$ of rationals.*

The first section contains the proof and discussion of Theorems 1 and 2, the second is devoted to the proof of Theorem 3, and the third deals with some unusual metrics in the group L of reals. This question has been suggested by the reasonings used in the second section.

1. Proof of Theorem 1. Assume to the contrary that for some sequence of elements and some integer s we have $x_n \to 0$ but $\|x_n/s\| > \delta > 0$ $(n = 1, 2, \ldots)$. We construct by induction a sequence $\{y_n\}$ such that

$$(1) \qquad \|y_n - y_{n-1}\| \leqslant \frac{\delta}{2^n},$$

$$(2) \qquad \left\|\frac{1}{s^i} y_n\right\| \geqslant \left(\frac{1}{4} + \frac{1}{2^n}\right) \quad \text{for} \quad i \leqslant n.$$

Put $y_1 = x_1$. If y_n is already defined, then if

$$\left\|\frac{y_n}{s^{n+1}}\right\| \geqslant \delta\left(\frac{1}{4} + \frac{1}{2^{n+1}}\right),$$

we put $y_{n+1} = y_n$ and satisfy in this way (1) and (2); in the opposite case we choose from the sequence $\{x_n\}$ a term x such that $\|x\| < \delta/2^{n+1}s^n$ and put $y_{n+1} = y_n + xs^n$. Then (1) is satisfied; moreover, if $i \leqslant n$, then

$$\left\|\frac{y_{n+1}}{s^i}\right\| = \left\|\frac{1}{s^i} y_n + xs^{n-i}\right\| \geqslant \left\|\frac{y_n}{s^i}\right\| - \|x\| s^{n-i} \geqslant \delta\left(\frac{1}{4} + \frac{1}{2^n}\right) - \frac{\delta}{2^{n+1} s^i}$$
$$\geqslant \delta\left(\frac{1}{4} + \frac{1}{2^{n+1}}\right).$$

It remains to check (2) for $i = n+1$:

$$\left\|\frac{y_{n+1}}{s^{n+1}}\right\| = \left\|\frac{y_n}{s^{n+1}} + \frac{x}{s}\right\| \geqslant \left\|\frac{x}{s}\right\| - \left\|\frac{y_n}{s^{n+1}}\right\| > \delta - \delta\left(\frac{1}{4} + \frac{1}{2^{n+1}}\right) > \delta\left(\frac{1}{4} + \frac{1}{2^{n+1}}\right).$$

On account of (1) and in view of the completeness of G there is a limit $y = \lim_n y_n$. For $i < n$ we have either $(y_{n+1} - y_n)/s^i = 0$ or $(y_{n+1} - y_n)/s^i = xs^{n-i}$. In both cases

$$\left\|\frac{1}{s^i}(y_{n+1} - y_n)\right\| < \frac{\delta}{2^n}.$$

Therefore for any i the sequence $\{y_n/s^i\}_{n=1,2,\ldots}$ is fundamental. Since G is torsion free its limit is equal to y/s^i.

From (2) we obtain $\|y_n/s^i\| \geqslant \delta/4$ for $i \leqslant n$ and so $\|y/s^i\| \geqslant \delta/4$ for every i which contradicts the assumption $y/n \to 0$.

Let now $\hat{R}$ denote the "solenoid", i. e., the dual group to the group R of rationals. There are two kinds of elements in $\hat{R}$: the "regular" ones which correspond to those characters of R which are continuous in the natural topology, and the "irregular", corresponding to non-continuous characters of R. If x is regular, it can be represented by an exponential: $x = x_\lambda(r) = e^{i\lambda r}$ (λ real, $r \in R$). According to [4], p. 177, every x is represented by a character defined for any prime p and any integer $m \geqslant 0$ as

$$(3) \qquad \chi\left(\frac{1}{p^m}\right) = \exp 2\pi i\left[\frac{1}{p^m}(\tau + k_{p,0} + k_{p,1}p + \ldots + k_{p,m-1}p^{m-1})\right],$$

where τ is a real number and $\{k_{p,m}\}$ a double sequence of integers such that $0 \leqslant k_{p,m} < p$.

LEMMA 1. *If $x \in \hat{R}$ and $\lim_n x/n = 0$, then x is regular.*

Proof. Obviously $\lim_n x/n = 0$ means $\lim_n \chi(1/n) = 1$ and $\lim_n \chi(1/p^m) = 1$ means that the expression in brackets in (3) tends to 0 or to 1 for $m \to \infty$. However, it is easily seen that this is impossible unless for m sufficiently large we have constantly $k_{p,m} = 0$ or constantly $k_{p,m} = p-1$. Both these cases lead to the issue that there is a real τ_p such that $\chi(1/p^m) = \exp(2\pi i\tau_p/p^m)$. It is to be proved that τ_p does not depend on p.

We take two primes, p and q. If $n = p^s q^t$, then

$$\frac{1}{n} = \frac{a}{q^t} - \frac{b}{p^s}$$

with $0 < a < q^t$, $(a, q) = 1$. We fix t. If $s \to \infty$, there is a number a_0 with $(a_0, q) = 1$ and an infinite sequence $\{s_k\}$ such that $1/p^{s_k}q^t = a_0/q^t - b_k/p^{s_k}$. We have

$$\lim_k \chi\left(\frac{1}{p^{s_k}q^t}\right) = \lim_k \exp 2\pi i\left(\frac{\tau_p b_k}{p^{s_k}} - \frac{\tau_q a_0}{q^t}\right) = 1$$

and since

$$\lim_k \frac{b^k}{p^{s_k}} = \frac{a_0}{q^t},$$

we have

$$\lim_k \exp 2\pi i\frac{\tau_p b_k}{p^{s_k}} = \exp 2\pi i\frac{\tau_p a_0}{q^t} = \exp 2\pi i\frac{\tau_q a_0}{q^t}.$$

Thus $(\tau_p-\tau_q)/q^t$ is an integer. Since t is arbitrary we have $\tau_p = \tau_q$ and the lemma is proved.

Proof of Theorem 2. Suppose first G is compact. The (discrete) character group $\hat{G}$ of G is divisible, since G is torsion free, and it is torsion free, since G is divisible (see e. g. [4], p. 164). But then G is a direct product of copies of the group R ([6], p. 149). Consequently, G is the Cartesian product of copies of $\hat{R}$. Thus for compact groups Theorem 2 can be reduced to the case $G = \hat{R}$.

Suppose $x \in \hat{R}$ is regular and $x = e^{i\lambda r}$; then $w_n x = e^{i\lambda w_n r}$. If $w_n \to 0$ we have $\lim_n e^{i\lambda w_n r} = 1$ for every r, which means that $w_n x$ tends to the identity element $x_0 = 1$ of R. If x is irregular, then $x/n \to 0$ fails to be true as is shown by Lemma 1, and so Theorem 2 is fulfilled in vacuo.

If G is generated by a compact neighbourhood of the identity, then it is the direct product of a compact group and a vector group ([7], 39). Since Theorem 2 holds for both factors (being trivial for the vector group) it holds equally for G.

Finally, every locally compact group contains an open subgroup $G_1 \subset G$ generated by a compact neighbourhood of the identity. Obviously $w_n x \to 0$ is possible solely for $x \in G_1$. Thus, Theorem 2 is proved.

The reader possibly observed that in order to obtain the announced positive answer to (Q) for locally compact groups Theorem 2 is "too strong". It would be enough to point out just *one* element $x \in \hat{R}$ for which x/n does not tend to the identity. An example of such element can be found in [4], p. 180. For the rest of the proof the group structure and duality arguments used in the proof of Theorem 2 would be sufficient.

2. No routine reasoning of this kind enables us to prove Theorem 3 and so to obtain the claimed general answer "no" to the problem (Q). Here there is a need for an example and the construction we will use rests essentially upon the next theorem.

THEOREM 4. *There is a function $N(r)$ defined for rationals and having the following properties*:

(i) $N(0) = 0$, $N(r) > 0$ *for* $r \neq 0$,

(ii) $N(r_1+r_2) \leqslant N(r_1)+N(r_2)$,

(iii) $N(-r) = N(r)$,

(iv) $N(wr_n) \to 0$ *if* $N(r_n) \to 0$ *for every rational* w,

(v) $N(kr) \geqslant N(r)$ *for every integer* $k > 0$,

(vi) $\lim_n N(r/n) = 0$,

(vii) $\limsup_{r\to 0} N(r) = \infty$.

Obviously property (vii) is "punctum saliens". Before we proceed to the proof itself we shall establish two lemmas.

LEMMA 2. *Let*

$$\frac{n}{m} = \sum_{i=1}^{s} \frac{\varepsilon_i}{a_i},$$

where n, m *and* a_i *are integers and* $\varepsilon_i = \pm 1$, $1 \leqslant a_1 \leqslant \ldots \leqslant a_s$, $n \neq 0$, $m > 0$. *If* t *is the least number for which*

$$\frac{n}{m} = \sum_{i=1}^{t} \frac{\varepsilon_i}{a_i} \tag{4}$$

(and so $t \leqslant s$*), then*

$$a_i \leqslant (mt)^{2^{i-1}} \quad \textit{for each } i \leqslant t. \tag{5}$$

Proof. We proceed by induction with respect to i. From (4) it follows

$$\left|\frac{n}{m}\right| \leqslant \sum_{i=1}^{t} \frac{1}{a_i} \leqslant \frac{t}{a_1};$$

hence $a_1 \leqslant tm/|n| \leqslant tm$. Let now (5) hold for $i = 1, 2, \ldots, j-1$ $(j \leqslant t)$. By the definition of t we have

$$\left|\frac{n}{m} - \sum_{i=1}^{j-1} \frac{\varepsilon_i}{a_i}\right| > 0;$$

hence

$$\left|\frac{n}{m} - \sum_{i=1}^{j-1} \frac{\varepsilon_i}{a_i}\right| \geqslant \frac{1}{ma_1 \ldots a_{j-1}} \geqslant \frac{1}{m} \prod_{i=1}^{j-1} (mt)^{-2^{i-1}} = \frac{1}{m} (mt)^{-2^{j-1}+1} \tag{6}$$

On the other hand,

$$\left|\frac{n}{m} - \sum_{i=1}^{j-1} \frac{\varepsilon_i}{a_i}\right| = \left|\sum_{i=j}^{t} \frac{\varepsilon_i}{a_i}\right| \leqslant \sum_{i=j}^{t} \frac{1}{a_i} \leqslant \frac{t}{a_j}. \tag{7}$$

From (6) and (7) we get $a_j \leqslant (mt)^{2^{j-1}}$, which ends the proof.

LEMMA 3. *Whatever be the integers* $j > 0$ *and* $k > j$, *the number* $\sum_{i=1}^{j} k^{-2^{i^2}}$ *cannot be written as* $\sum_{i=1}^{s} (\varepsilon_i/a_i)$ *with* $s < j$.

The proof will be by induction with respect to j. For $j = 1$ there is nothing to be proved. We suppose that the assertion holds for $j = 1, 2, \ldots, p-1$ $(2 \leqslant p < k-1)$. Assume to the contrary that

$$T_{k,p} = \sum_{i=1}^{p} k^{-2^{i^2}} = \sum_{i=1}^{r} \frac{\varepsilon_i}{a_i} \quad \text{with} \quad r < p.$$

Then

$$\sum_{i=1}^{p-1} k^{-2^{i^2}} = T_{k,p-1} = \sum_{i=1}^{r} \frac{\varepsilon_i}{a_i} - \frac{1}{k^{2^{p^2}}}. \tag{8}$$

Thus, $T_{k,p-1}$ is represented as the sum of $r+1$ non-zero terms. Hence no sum of r terms among them can be equal to $T_{k,p-1}$. Moreover, since $r \leqslant p-1$, the inductive assumption forbids any sum of less than r terms to equal $T_{k,p-1}$. So we can apply Lemma 2 to the second equation in (8) putting $m = k^{2^{(p-1)^2}}$, $t = s = r+1$, $a_{r+1} = k^{2^{p^2}}$. Taking in account that $r \leqslant p-1 < k$ we obtain from (5) for $i = r+1$ the inequalities

$$k^{2^{p^2}} \leqslant [k^{2^{(p-1)^2}}(r+1)]^{2^r} \leqslant (k^{2^{(p-1)^2}+1})^{2^{p-1}} < k^{2^{p^2}}.$$

This contradiction achieves the proof.

Proof of Theorem 4. We put

$$N(r) = \inf \sum_{i=1}^{s} (\log\log a_i e^2)^{-1/3},$$

where the greatest lower bound is taken with respect to all representations of r in the form $\sum_{i=1}^{s} (\varepsilon_i/a_i)$. It is obvious that

$$|r| \leqslant N(r). \tag{9}$$

This yields property (i). Properties (ii)-(vi) are clearly satisfied. One has but to prove (vii).

If

$$r_k = \sum_{i=1}^{k} k^{-2^{i^2}} = \sum_{i=1}^{s} \frac{\varepsilon_i}{a_i} \quad (a_1 \leqslant a_2 \leqslant \ldots \leqslant a_s),$$

then on account of Lemma 2 we have for some $t \leqslant s$ the equation

$$r_k = \sum_{i=1}^{k} \frac{\varepsilon_i}{a_i}, \quad a_i < (k^{2^{k^2}} t)^{2^{t-1}} \quad (1 \leqslant i \leqslant t).$$

By Lemma 3 we have $t \geqslant k$. Hence

$$a_i \leqslant (t^{2^{t^2}+1})^{2^{t-1}}.$$

Choosing an $a > 0$ so as to have $(1+a)\log 2 < 1$ we get

$$a_i \leqslant t^{2^{(1+a)t^2}} = O(\exp\exp t^2).$$

Hence $(\log\log a_i e^2)^{1/3} = O(t^{2/3})$ and finally, as $t \leqslant s$,

$$\sum_{i=1}^{s} (\log\log a_i e^2)^{-1/3} \geqslant \sum_{i=1}^{t} (\log\log a_i e^2)^{-1/3} \geqslant \frac{t}{O(t^{2/3})} = \frac{t^{1/3}}{O(1)}.$$

Thus, we have for any representation of r_k in the form $\sum_{i=1}^{s}(\varepsilon_i/a_i)$ and for a suitable t the inequalities

$$\sum_{i=1}^{s} (\log\log a_i e^2)^{-1/3} \geqslant ct^{1/3} \geqslant ck^{1/3} \quad \text{where} \quad c > 0.$$

Therefore $N(r_k) \geqslant ck^{1/3}$. Since $\lim_k r_k = 0$, property (vii) is fulfilled.

Proof of Theorem 3. On account of Theorem 4 the group R with the (unusual) metric $\varrho(r_1, r_2) = N(r_1 - r_2)$ fulfills all requirements of Theorem 3 except completeness. We enlarge it to a complete metric group Γ by applying the classical Cantor process and extending the group operation by continuity. Obviously Γ is separable. From (v) it follows that Γ is torsion free. Owing to (iv) it is also divisible. In fact, if a sequence $\{r_n\}$, $r_n \epsilon R$, is fundamental in ϱ, then, for every integer k, $\{r_n/k\}$ is fundamental, too. Conditions (v) and (vi) imply $\lim_n N(x/n) = 0$ for every $x \epsilon \Gamma$. Finally, (vii) implies the existence of an x (e. g. $x = 1$) and a sequence $r_n \to 0$ such that $N(r_n x)$ does not tend to zero. Thus, Γ has all desired properties.

3. From (9) it follows that a sequence of rationals which is fundamental in ϱ is also fundamental in the natural metric. Hence, to every element $x \epsilon \Gamma$ (with $\lim_n \varrho(x, r_n) = 0$, say) there corresponds a real number $\varphi(x)$ (ordinary limit of $\{r_n\}$). This correspondence is a continuous homomorphism of Γ onto a subgroup Γ_1 of the group L of reals.

A modification and actually a simplification of the construction described in Section 2 may be studied in order to exhibit another phenomenon, namely an effectively defined complete metric in the group of reals which is neither discrete nor equivalent to the natural metric. If S is the group of the dyadic rationals, $r \epsilon S$ and

$$r = \sum_{n=1}^{k} \frac{\delta_n}{2^n} \quad (k < \infty,\ \delta_n = 0 \text{ or } 1)$$

we put

$$D(r) = \sum_{n=1}^{k} \frac{\delta_n}{n+1}.$$

For other $r \epsilon S$ we define $D(r) = 1$ if $r \geqslant 1$ and $D(-r) = D(r)$. The "norm" thus obtained in the group S fulfils obviously conditions (i) and (iii), where D is substituted for N. The triangle inequality (ii) needs a verification, which, however, reduces to a simple arithmetical computation we omit here. But also condition (vii) is satisfied, because

$$r_k = \sum_{n=k}^{k^2} \frac{1}{2^n} \to 0 \quad \text{and} \quad D(r_k) \to \infty.$$

Like for the norm N we construct a complete group T starting from S and applying the Cantor process. This is not a divisible group but it is not the point we are now interested in. We rather emphasize that D fulfils (9) for $|r| < 1$ and hence a continuous homorphism $\psi\colon T \to L$ can be defined with respect to D just in that way as φ was defined with respect to N. If H denotes the kernel $\psi^{-1}(0) \subset T$ of ψ, then T/H is algebraically isomorphic to $T_1 = \psi(T)$. If $x \in T$ and $x^* = x+H \in T/H$, we put

$$D^*(x^*) = \inf_{y \in H} D(x+y).$$

With this norm and with the metric $\varrho^*(x^*, y^*) = D^*(x^*-y^*)$ the group T_1 becomes a complete metric group. The proof is a routine.

The norm D^* induces a norm in $\psi(T)$ by isomorphism. We can extend it to the whole of L by putting $\tilde{D}(a) = D^*(a)$ for $a \in \psi(T)$ and $\tilde{D}(a) = 1$ for $a \in L \setminus T_1$. It is obvious that L with the (invariant) metric $\tilde{\varrho}(a, \beta) = \tilde{D}(a-\beta)$ becomes a complete metric group. The metric $\tilde{\varrho}$ was constructed quite effectively. T_1 is a Borel set and $D^*(a)$ and so $\tilde{D}(a)$ are B-measurable functions. In fact, $a \in T_1$ means that there are sequences $\{r_k\}$, $r_k \in S$, fundamental in D and such that $r_k \to a$ (in the usual sense). But if such sequences exist and if

$$a = \sum_{n=1}^{\infty} \frac{\delta_n}{2^n}$$

is irrational, then

$$r_k = \sum_{n=1}^{k} \frac{\delta_n}{2^n}$$

is one of them as follows from the definition of the norm D and from the fact that for every k the digits δ_n of dyadic rationals, sufficiently close to a, coincide with those of a for $n = 1, \ldots, k$. So $x \in T_1$ is equivalent with $\sum(\delta_n/n) < \infty$. The same remarks and the definition of D^* lead to the conclusion that for $a \in T_1$ we have $D^*(a) = \sum_n (\delta_n/n)$.

However, the metric $\tilde{\varrho}$ is not topologically equivalent with the ordinary metric, since in view of $T_1 \neq L$ it is not separable. Thus we have proved

THEOREM 5. *In the group L of reals an invariant metric ϱ can be effectively constructed so that L is complete, non-discrete and non-separable. The function $\varrho(a, 0)$ is a B-measurable function of a.*

Remark. Theorem 5 could also be proved starting from the norm N, introduced in section 2, and using the group Γ_1 instead of T_1. However, the reasoning would be much more complicated. The essential difficulty consists in proving that Γ_1 does not equal L, i. e., that there are numbers not of the form $\varphi(x)$, $x \in \Gamma$. We are indebted to P. Erdős for having accomplished this proof by showing that Γ_1 consists of Liouville numbers only. This result, quoted here with its author's consent, seems interesting for itself. Herefrom can be easily deduced, by a purely topological argument, that φ is not one-to-one.

We notice that a separable complete Borel metric in L must be topologically equivalent with the usual metric. Here is the reason for it: if $\varrho^*(a, 0)$ is B-measurable, then $U \subset L$ denoting a ϱ^*-open set and i the identity map of L under the usual topology onto L^* (i. e., L metrized by ϱ^*) the set $i^{-1}(U)$ is Borel in L. If L^* is separable, then according to a theorem of Kuratowski ([5], p. 306) i fulfils the Baire condition, i. e., it is continuous on a set $L \setminus Z$, where Z is a set of the first category. But then i, being an isomorphism of one complete group onto another, must be continuous ([2], p. 23), and, since L is separable, i is bicontinuous [1].

On the other hand, there are separable invariant non-natural metrics in L but they are either not complete or not effective. An example of the first kind is the precompact topology generated by an almost periodic non-periodic function. An example of the second kind is the compact topology transferred to L from the solenoid $\hat{R}$ by an algebraic isomorphism (compare [3] and [4], Chapt. VI).

We finish by pointing out another peculiar phenomenon, which can occur in metric divisible torsion free Abelian groups. It can namely happen that x/n does not tend to the identity for any x, but comes arbitrarily near to the identity for every x. To see this, take a direct factor B of $\hat{R}$, complementary to the (divisible) subgroup consisting of all regular elements of $\hat{R}$ (comp. section 1). According to Lemma 1, x/n does not tend to the identity if $x \in B$. On the other hand, for every $x \in \hat{R}$ the set $\{x/n\}$ has the identity as a cluster point, this being true for all compact divisible groups ([4], p. 176).

It can be proved by standard methods used in section 1 that the phenomenon just described cannot occur in locally compact torsion free Abelian groups. We do not know whether it can occur in complete groups (**P 498**).

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